Given: $f(x) = \cos(x)$

a. Determine a **line** $P(x)$ that closely matches the characteristics of $f(x)$ at the point $(0, 1)$.

$P(x) =$ 1

b. Use Geogebra to visualize how $f(x)$ and $P(x)$ interact.

c. Fill in the following table (rounding to 3 decimal places when necessary).

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	.995	1	.005
.5	.87758	1	.12242
1	.5403	1	.4597

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$f'(0) = 0$

$y = 1$

$P(x) = 1 + 0x$

$p(x) = ax + b$

$p'(x) = a$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

Given: $f(x) = \cos(x)$

a. Determine a **parabola** $P(x)$ that closely matches the characteristics of $f(x)$ at the point $(0, 1)$.

$P(x) =$ $-\frac{1}{2}x^2 + 1$

$p(x) = 1 + 0x - \frac{1}{2}x^2$

b. Use Geogebra to visualize how $f(x)$ and $P(x)$ interact.

c. Fill in the following table (rounding to 3 decimal places when necessary).

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	.995	.995	0
.5	.87758	.875	.00258
1	.5403	.5	.0403

$p(x) = ax^2 + bx + c$

$p(0) = 1$

$p'(x) = 2ax + b$

$p'(0) = 0$

$p''(x) = 2a$

$p''(0) = -1$

$2a = -1$ $(0,1)$

$a = -\frac{1}{2}$

$2ax + b = 0$

$2(-\frac{1}{2})(0) + b = 0$

$b = 0$

$ax^2 + bx + c = 1$

$-\frac{1}{2}(0)^2 + 0 + c = 1$

$c = 1$

$f(x) = \cos(x)$

$f(0) = 1$

$f'(x) = -\sin(x)$

$f'(0) = 0$

$f''(x) = -\cos(x)$

$f''(0) = -1$

Given: $g(x) = e^x$

a. Determine a **line** $P(x)$ that closely matches the characteristics of $f(x)$ at the point $(0, 1)$.

$P(x) =$ $x + 1$

b. Use Geogebra to visualize how $f(x)$ and $P(x)$ interact.

c. Fill in the following table (rounding to 3 decimal places when necessary).

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	1.1052	1.1	.00517
.5	1.6487	1.5	.14872
1	2.7183	2	.71828

$p(x) = ax + b$

$p(0) = 1$

$ax + b = 1$

$(1)(0) + b = 1$

$b = 1$

$f(x) = e^x$

$f(0) = 1$

$p'(x) = a$

$p'(0) = 1$

$a = 1$

$f'(x) = e^x$

$f'(0) = 1$

$y = 1x + 1$

Given: $g(x) = e^x$

a. Determine a **parabola** $P(x)$ that closely matches the characteristics of $f(x)$ at the point $(0, 1)$.

$P(x) =$ $\frac{1}{2}x^2 + x + 1$

b. Use Geogebra to visualize how $f(x)$ and $P(x)$ interact.

c. Fill in the following table (rounding to 3 decimal places when necessary).

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0.
.1	1.1052	1.105	0
.5	1.6487	1.625	.02372
1	2.7183	2.5	.21828

$p(x) = ax^2 + bx + c$

$p(0) = 1$

$2a = 1$

$a = \frac{1}{2}$

$f(x) = e^x$

$f(0) = 1$

$p'(x) = 2ax + b$

$p'(0) = 1$

$b = 1$

$f'(x) = e^x$

$f'(0) = 1$

$c = 1$

$p''(x) = 2a$

$p''(0) = 1$

$f''(x) = e^x$

$f''(0) = 1$

Given: $f(x) = \cos(x)$

Determine a **fourth degree polynomial**, $P(x)$, that closely matches the characteristics of $f(x)$ when centered at $x = 0$.

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$P'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$P''(x) = 12ax^2 + 6bx + 2c$

$P'''(x) = 24ax + 6b$

$P^{(4)}(x) = 24a$

$24a = 1$ $\quad a = \frac{1}{24}$

$6b = 0$ $\quad b = 0$

$2c = -1$ $\quad c = \frac{-1}{2}$

$d = 0$

$e = 1$

$f(x) = \cos(x)$

$f(0) = 1$

$f'(x) = -\sin(x)$

$f'(0) = 0$

$f''(x) = -\cos(x)$

$f''(0) = -1$

$f'''(x) = \sin(x)$

$f'''(0) = 0$

$f^{(4)}(x) = \cos(x)$

$f^{(4)}(0) = 1$

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	.995	.995	0
.5	.87758	.8776	.000022
1	.5403	.54167	.00136

$P(x) = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{24}x^4$ $\quad P(x) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$

Given: $g(x) = e^x$

Determine a **fourth degree polynomial**, $P(x)$, that closely matches the characteristics of $f(x)$ when centered at $x = 0$.

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$P'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$P''(x) = 12ax^2 + 6bx + 2c$

$P'''(x) = 24ax + 6b$

$P^{(4)}(x) = 24a$

$24a = 1$ $\quad a = \frac{1}{24}$

$6b = 1$ $\quad b = \frac{1}{6}$

$2c = 1$ $\quad c = \frac{1}{2}$

$d = 1$

$e = 1$

$P(x) = \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	1.1052	1.1052	0
.5	1.6487	1.6484	.00028
1	2.7183	2.7083	.00995

CENTERED AT a

Taylor Series as an infinite sum. $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Write a Taylor/**Maclaurin (centered at $x = 0$)** Series polynomial function with at least 4 terms for each of the following functions. Then write the polynomial function as an infinite series. Use a separate sheet of paper for work.

1. $f(x) = \cos(x)$	$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
2. $f(x) = e^x$	$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
3. $f(x) = \ln(1+x)$	$0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$
4. $f(x) = \sin(x)$		$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
5. $f(x) = \frac{1}{1-x}$		$\displaystyle\sum_{n=0}^{\infty} x^n$
6. $f(x) = tan^{-1}x$		$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

Investigating the Taylor/MacLaurin Series for $f(x) = tan^{-1}x$ centered at $x = 0$.

$f(x) = \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ $(1-x)^{-1}$	$g(x) = tan^{-1}x = \displaystyle\sum_{n=0}^{\infty}$
$f'(x) = -1(1-x)^{-2} = \dfrac{-1}{(1-x)^2} =$	$g'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(\quad)} = \displaystyle\sum_{n=0}^{\infty}$

Taylor Series as an infinite sum.

Write a Taylor/Maclaurin (centered at $x = 0$) Series polynomial function with at least 4 terms for each of the following functions. Then write the polynomial function as an infinite series. Use a separate sheet of paper for work.

1. $f(x) = \cos(x)$	$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
2. $f(x) = e^x$	$P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
3. $f(x) = \ln(1+x)$	$P(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$
4. $f(x) = \sin(x)$	$P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
5. $f(x) = \frac{1}{1-x}$	$P(x) = 1 + x + x^2 + x^3 + \ldots$	$\sum_{n=0}^{\infty} x^n$
6. $f(x) = tan^{-1}x$	$P(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$\sum_{n=0}^{\infty} r^n = \frac{a_1}{1-r}$

$y = x^n$

$\frac{1}{2n+1} \cdot x^{2n+1}$

Investigating the Taylor/MacLaurin Series for $f(x) = tan^{-1}x$ centered at $x = 0$.

$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$g(x) = tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
$f'(x) = -1(1-x)^{-2} = \frac{-1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$	$g'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$

Power Series and Radius of Convergence

A Power Series centered at $x = a$ is a function of x and can be written as:

$$f(x) = C_0 + C_1(x-a)^1 + C_2(x-a)^2 + C_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} C_n(x-a)^n$$

* For each Power Series, a. Write out the first 4 terms of the series, then b. Use the Ratio Test to determine both the Interval of Convergence and Radius of Convergence.

1. $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ $= 1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \cdots$

RATIO TEST

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n (x-0)^n$$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{x}{2}\right)^{n+1}}{\left(\frac{x}{2}\right)^n} \right| = \lim_{n\to\infty} \left| \frac{\left(\frac{x}{2}\right)^n\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^n} \right| = \left|\frac{x}{2}\right|$$

TO BE CONVERGENT $\left|\frac{x}{2}\right| < 1 \longrightarrow -1 < \frac{x}{2} < 1$

CENTERED AT $x = 0$

$-2 < x < 2$

INTERVAL OF CONVERGENCE

RADIUS OF CONVERGENCE IS 2

2. $\sum_{n=0}^{\infty} \frac{n^3(x+5)^n}{6^n}$ $= 0 + \frac{x+5}{6} + \frac{2^3(x+5)^2}{6^2} + \frac{3^3(x+5)^3}{6^3}$

$$\lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \frac{(x+5)^{n+1}}{(x+5)^n} \frac{6^n}{6^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \frac{(x+5)^n(x+5)}{(x+5)^n} \frac{6^n}{6^n \cdot 6} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \frac{(x+5)}{6} \right| = \left| \frac{x+5}{6} \right|$$

TO BE CONVERGENT $\left|\frac{x+5}{6}\right| < 1 \longrightarrow -1 < \frac{x+5}{6} < 1$

$-6 < x+5 < 6$

CENTERED AT $x = -5$

$-11 < x < 1$

RADIUS OF CONV IS 6.

INTERVAL OF CONVERGENCE

TEST ENDPOINTS

$x = -11$

$$\sum_{n=0}^{\infty} \frac{n^3(-6)^n}{6^n} = \sum_{n=0}^{\infty} (-1)^n \cdot n^3$$

DIVERGENT

$x = 1$

$$\sum_{n=0}^{\infty} n^3 \frac{(6)^n}{(6)^n} = \sum_{n=0}^{\infty} n^3$$

DIVERGENT

3. $\sum_{n=0}^{\infty} \frac{(-1)^n(n)}{4^n}(x+3)^n$ $= 0 + \frac{-1}{4}(x+3) + \frac{2}{4^2}(x+3)^2 + \frac{-3}{4^3}(x+3)^3$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{(n+1)}{n} \frac{4^n}{4^{n+1}} \frac{(x+3)^{n+1}}{(x+3)} \right| = \left| \frac{(-1)\ (x+3)}{4} \right|$$

TO BE CONVERGENT $\left| \frac{(-1)(x+3)}{4} \right| < 1 \longrightarrow -1 < \frac{x+3}{4} < 1$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$

CENTERED AT $x = -3$

RADIUS IS 4.

$-4 < x+3 < 4$

$-7 < x < 1$

INTERVAL OF CONVERGENCE

4. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{x^n \cdot x \cdot n!}{x^n \cdot (n+1)\cdot n!} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$$

5. $\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$

3. $\sum_{n=0}^{\infty} \frac{(-1)^n(n)}{4^n}(x+3)^n$ $= 0 - \frac{1}{4}(x+3) + \frac{1}{8}(x+3)^2 - \frac{3}{4^3}(x+3)^3 + \cdots$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)}{4^{n+1}}(x+3)^{n+1} \cdot \frac{4^n}{(x+3)^n\, n\, (-1)^n} \right| = \left| \frac{(-1)(x+3)}{4} \right|$$

$$-1 < \frac{(-1)(x+3)}{4} < 1$$

$$-4 < (-1)(x+3) < 4$$

$$4 > x+3 > -4$$

$$-7 < x < 1$$

RADIUS IS 4

4. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

IN CLASS NOTES

5. $\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$ $= \sum_{n=1}^{\infty} \frac{2^n}{n}(4(x-2))^n = \sum_{n=1}^{\infty} \frac{8^n}{n}(x-2)^n$

$$= \frac{8}{1}(x-2)^1 + \frac{64}{2}(x-2)^2 + \frac{8^3}{3}(x-2)^3 + \frac{8^4}{4}(x-2)^4$$

$$\lim_{n\to\infty} \left| \frac{8^{n+1}}{n+1}(x-2)^{n+1} \cdot \frac{n}{8^n \cdot (x-2)^n} \right| = \lim_{n\to\infty} |8(x-2)|$$

$$\frac{2^n}{n}\left(\frac{15}{2}-8\right)^n \qquad \frac{15}{8} \leq x \; \frac{17}{8} \qquad \frac{2^n}{n}\left(\frac{17}{2}-8\right)$$

$$\frac{(-1)^n}{n} \qquad \frac{(1)^n}{n}$$

IN CLASS NOTES

6. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$ $= 1 - \frac{1}{3}x^2 + \frac{1}{3^2}x^4 - \frac{1}{3^3}x^6$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(-3)^{n+1}} \frac{(-3)^n}{x^{2n}} \right| = \left| \frac{x^2}{(-3)} \right|$$

$$-1 < \frac{x^2}{-3} < 1$$

$$3 > x^2 > -3$$

$$\sqrt{3} > x > -\sqrt{3}$$

7. $\sum_{n=0}^{\infty} n!\,(x-2)^n$ $= 1 + (x-2) + 2(x-2)^2 + 6(x-2)^3 + \ldots$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n} \right| = \lim_{n\to\infty} \left| (n+1)(x-2) \right| = \infty$$

NEVER CONVERGES

NO INTERVAL OF CONVERGENCE

8) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (x-1)^n$ $= (x-1) + -\frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}\,(x-1)^{n+1}}{(n+1)} \cdot \frac{n}{(-1)^{n+1}(x-1)^n} \right| = \left| \frac{(-1)(x-1)}{} \right| = |x-1|$$

$$-1 < x-1 < 1$$

$$0 < -x \leq 2$$

IF $x = 0$

$$\frac{(-1)^n(-1)\;(-1)^n}{n} = \frac{-1}{n}$$ DIVERGENT

IF $x = 2$

$$\frac{(-1)^n(-1)\;(1)^n}{n} = \frac{(-1)^{n+1}}{n}$$