Taylor/MacLaurin Series to Know

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \ldots = \sum_{n=0}^{\infty} x^n$

Function Value	Taylor Series Estimate (first 2 terms) include Error and $\lvert 3rd\ term \rvert$	Taylor Series Estimate (first 3 terms) include Error and $\lvert 4th\ term \rvert$	Taylor Series Estimate (first 4 terms) include Error and $\lvert 5th\ term \rvert$
$\cos(0.5)$	ESTIMATE $= 1 - \frac{(\frac{1}{2})^2}{2!} = \frac{7}{8}$ THIRD TERM $= \frac{(\frac{1}{2})^4}{4!} = .002604$ ERROR $= \lvert \cos(\frac{1}{2}) - \frac{7}{8} \rvert = .00258$	ESTIMATE $\approx .877604$ FOURTH TERM $\approx .0000217014$ ERROR $\approx .000021605$	ESTIMATE $\approx .8775825$ FIFTH TERM $\approx .000000968812$ ERROR $\approx .000000966126$
$\sin(0.75)$	ESTIMATE $\approx .6796875$ THIRD TERM $\approx .0019775391$ ERROR $\approx .00195126$	ESTIMATE $\approx .68166504$ FOURTH TERM $\approx .000026485$ ~~.0002471923~~ ERROR $\approx .000026279$	ESTIMATE $\approx .68163855$ FIFTH TERM $\approx .000000206913$ ERROR $\approx .000000205859$

WHEN ASK TO ESTIMATE A FUNCTION USING THE n^{TH} TERM POLY.

$\lvert ERROR \rvert \leq \lvert a_{n+1} \rvert$ "ALTERNATING SERIES ERROR BOUND"

Absolute vs Conditional Convergence

A series $\sum_{n=0}^{\infty} a_n$ **converges absolutely** if the absolute value series $\sum_{n=0}^{\infty} \lvert a_n \rvert$ converges.
A series $\sum_{n=0}^{\infty} a_n$ **converges conditionally** if the absolute value series $\sum_{n=0}^{\infty} \lvert a_n \rvert$ diverges, but $\sum_{n=0}^{\infty} a_n$ converges.

For our purposes, compare the following alternating series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ A.S.T. CONVERGES

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n^1}$$ DIVERGES BECAUSE HARMONIC

CONVERGES CONDITIONALLY

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$ CONVERGES BECAUSE OF A.S.T.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ CONVERGES B/C THIS IS A p-SERIES w/ p > 1

CONVERGES ABSOLUTELY

Example: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{2n}}{n!}$ SERIES CONVERGES B/C LIMIT < 1

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(-1)^{n+1}} \frac{3^{2(n+1)}}{3^{2n}} \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\, 3^2}{(n+1)} \right| = 0$$

$\sum_{n=1}^{\infty} \frac{3^{2n}}{n!}$ L = 0

SINCE L < 1 SERIES CONVERGES

∴ SERIES IS ABSOLUTELY CONVERGES

Alternating Series Error Bound

For each question below:

a. find an approximation to the sum of the infinite series using the indicated number of terms.

b. set up an inequality to determine the maximum error for your approximation. Find this maximum error.

c. use your answer from part (b) to find an interval where the sum of the infinite series must exist.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n)}{2^n}$, using three terms

a. $\approx \frac{(+1)(1)}{2^1} + \frac{(-1)(2)}{2^2} + \frac{(+1)(3)}{2^3} = \frac{3}{8}$

b. $\lvert \text{ERROR} \rvert \le \lvert a_4 \rvert$ $\quad \lvert \text{ERROR} \rvert \le \frac{1}{4}$

$-\frac{1}{4} \le \text{ERROR} \le \frac{1}{4}$

c. $\frac{1}{8} \le f(x) \le \frac{5}{8}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3)}{n^2}$ using six terms

$\approx \frac{3}{1} - \frac{3}{4} + \frac{3}{9} - \frac{3}{16} + \frac{3}{25} - \frac{3}{36} = 2.4325$

b. $\lvert \text{ERROR} \rvert \le \lvert a_7 \rvert$ $\quad \lvert E \rvert \le \frac{3}{49}$

c. $2.37128 \le f(x) \le 2.49372$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ using five terms

a. $\approx -1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}$
$= -\frac{19}{30}$

b. $\lvert E \rvert \le \lvert \frac{1}{720} \rvert$

c. $\frac{-457}{720} \le f(x) \le \frac{-91}{144}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ using four terms

a. $\approx -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = -\frac{7}{12}$

b. $\lvert \text{ERROR} \rvert \le \frac{1}{5}$

c. $-\frac{7}{12} - \frac{1}{5} \le f(x) \le -\frac{7}{12} + \frac{1}{5}$

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{n}x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

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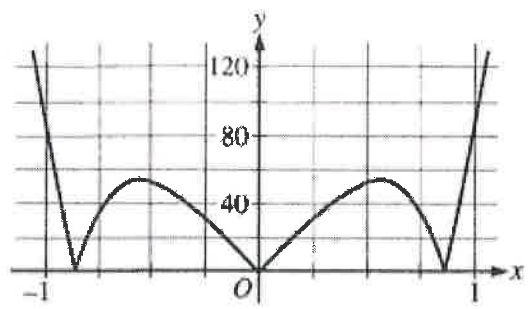
A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)\, dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

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$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

✓ **The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.**

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

No Calculators here.

1. Given $f(x) = \frac{1}{1-x}$, approximate $f(0.1)$ using a second degree MacLaurin Polynomial and find the error.

$f(.1) = \frac{1}{\frac{9}{10}} = \frac{10}{9} \approx 1.\bar{1}$

$P(x) = 1 + x + x^2$

$P(.1) = 1.11$

$\text{Error} = |P(.1) - f(.1)| = .00\bar{1}$

2. Find the error bound involved in calculating the sum of the first six terms of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

$1 + -1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}$

$|\text{Error}| \le \frac{1}{6!}$

Taylor's Remainder Theorem or Lagrange Error Bound

$$|R_n| \le \frac{f^{n+1}(z)|x-a|^{n+1}}{(n+1)!}$$

(Compare w/ Taylor Polynomial Formula)

$f^{n+1}(z)$ is the maximum value between a and x by looking at the next unused term. (looking at the $n+1$ derivative of $f(x)$)

3. Find the error bound for $f(x) = \frac{1}{1-x}$, using a second degree McLaurin Polynomial at $f(0.1)$.

$|\text{Error}| \le \frac{f^{(3)}(z)\,|(0-.1)^{3}|}{3!} = \cancel{\frac{6\left(\frac{1}{10}\right)^3}{6}} = \frac{1}{1000}$

$= .001$

$f(x) = (1-x)^{-1}$

$f'(x) = -1(1-x)^{-2}(-1)$

$= (1-x)^{-2}$

$f''(x) = -2(1-x)^{-3} \cdot -1$

$= 2(1-x)^{-3}$

$f'''(x) = 6(1-x)^{-4} = \frac{6}{(1-x)^4}$

$= \cancel{\frac{6\left(\frac{10}{9}\right)^4\left(\frac{1}{10}\right)^3}{6}}$

$\frac{\cancel{6}\left(\frac{10}{9}\right)^4\left(\frac{1}{10}\right)^3}{\cancel{6}} = \frac{10}{9^4} \quad \frac{10}{81 \cdot 81} = .0015$

$f'''(0) = 6$

$f'''(.1) = \frac{6}{\left(\frac{9}{10}\right)^4}$

You can now use calculators here.

4. Write a fourth-degree Maclaurin polynomial for $f(x) = e^x$. Then use your polynomial to approximate e^{-1}. Approximate the error bound for the maximum error for this approximation.

$\approx .367879$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}$$

5. Find the fourth-degree Taylor polynomial for $\cos x$ about $x = 0$. Then use your polynomial to approximate the value of $\cos 0.8$, and determine the error bound for the maximum error of this approximation.

$\approx .696707$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$1 - \frac{(.8)^2}{2} + \frac{(.8)^4}{24} \approx .69707$$

$$\text{ERROR} \le \left|\frac{-(.8)^6}{6!}\right| = \frac{.8^6}{6!}$$

$\approx .0003641$

6. Find the radius and interval of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{(x-2)^{n+1}}{(x-2)^n} \frac{3^n}{3^{n+1}} \frac{(n)^2}{(n+1)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(x-2)}{3} \right| < 1$$

$$-1 < \frac{x-2}{3} < 1$$

$$-3 < x - 2 < 3$$

$$-1 \le x \le 5$$

WHEN $x = -1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{n^2 (3)^n} = \sum_{n=0}^{\infty} \frac{1}{n^2}$$

CONVERGES

WHEN $x = 5$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3)^n}{(3)^n n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

CONVERGES

7. Let f be the function defined by $f(x) = \sqrt{x}$.
a. Find the second-degree Taylor polynomial about $x = 4$ for the function f.
b. Use your answer to estimate the value of $f(4.2)$.
c. Find a bound on the error for the approximation in part b.

$$P(x) = \frac{2(x-4)^0}{0!} + \frac{1}{4}\frac{(x-4)^1}{1} - \frac{(x-4)^2}{32(2)}$$

$$= 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64}$$

$f(x) = \sqrt{x} = x^{1/2}$ $\quad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $\quad f'(4) = \frac{1}{4}$

$f''(x) = \frac{-1}{4}x^{-3/2} = \frac{-1}{4\sqrt{x^3}}$ $\quad f''(4) = \frac{-1}{32}$

$f'''(x) = \frac{3}{8}x^{-5/2}$ $\quad f^{(3)}(4) = \frac{3}{8 \cdot 32}$

b. $f(4.2) \approx 2.0494$

$$\text{ERROR} \le \frac{\frac{3}{8 \cdot 32}(.2)^3}{6} = \frac{3 \cdot (.2)^3}{8 \cdot 6 \cdot 32}$$

$\approx .0000156$

8. Calculator permitted.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

a. $h(x) \approx \frac{80\,(x-2)^0}{0!} + \frac{128\,(x-2)^1}{1!}$

$h(1.9) \approx 80 + 128\left(\frac{-1}{10}\right) = 80 - 12.8 = 67.2$

LINEAR APPROX OF 67.2 IS AN UNDER APPROXIMATION

BECAUSE $h''(1.9) > 0$.

b. $h(x) \approx 80 + 128(x-2) + \frac{488}{3 \cdot 2!}(x-2)^2 + \frac{448}{3 \cdot 3!}(x-2)^3$

$h(1.9) \approx 80 - 12.8 + \frac{488}{6}\left(\frac{1}{100}\right) + \frac{448}{18}\left(\frac{-1}{1000}\right)$

$\approx 67.2 + \frac{488}{6 \cdot 100} - \frac{448}{18\,(+1000)} = 67.988\overline{4}$

c. $\left| h(1.9) - 67.988\overline{4} \right| \le \frac{584}{9 \cdot 4!}(1.9-2)^4 = 2.\overline{703} \times 10^{-4} < 3 \times 10^{-4}$

max

9. Calculator not permitted.

The Taylor series about $x=3$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x=3$ is given by

$$f^{(n)}(3)=\frac{(-1)^n\, n!}{5^n\,(n+3)} \quad \text{and} \quad f(3)=\frac{1}{3}$$

(a) Write the fourth-degree Taylor polynomial for f about $x=3$.

$$f(x) \approx \frac{\frac{1}{3}}{0!}(x-3)^0 + \frac{(-1)\,1!}{5(4)\cdot 1!}(x-3)^1 + \frac{2!}{5^2(5)\,2!}(x-3)^2$$

$$-\frac{1}{5^3(6)}(x-3)^3 + \frac{1}{5^4(7)}(x-3)^4$$

$$\frac{(-1)^n\,(x-3)^n}{5^n\,(n+3)}$$

(b) Find the radius of convergence of the Taylor series for f about $x=3$.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\; 5^n\; (x-3)^{n+1}\; (n+3)}{(-1)^n\; 5^{n+1}\; (x-3)^n\; (n+4)}\right| = \frac{x-3}{5}$$

RADIUS OF CONVERGENCE IS 5

$125 \times 6 = 750$ $\quad 5\,)\overline{750} = 150$

(c) Show that the third-degree Taylor polynomial approximates $f(4)$ with an error less than $\frac{1}{4000}$.

$$\cancel{P_3(4) = \frac{1}{3} - \frac{1}{20} + \frac{1}{125} - \frac{1}{750}}$$

$$\cancel{= \frac{17}{60} + \frac{5}{750} = \frac{17}{60} + \frac{2}{300} = \frac{87}{300}}$$

$$\left| f(4) - \overset{P_3(4)}{\cancel{\tfrac{87}{300}}} \right| \le \frac{1}{625\cdot 7} < \frac{1}{4000}$$