1. Basic population growth/decay says that the rate of change of the population P is proportional to the population itself.

a. Discuss why this intuitively makes sense.
b. Symbolize this proportion and then write an equation relating the values in this proportion.
c. Derive an equation relating the Population to the Time when $P(0) = P_0$.
d. The population of a particular city is growing at a rate proportional to its population. If the growth rate per year is 4% of the current population, how long will it take for the population to increase by 50%?
e. Cellium-314 decays at a rate proportional to the quantity present. Its half-life is 1612 years. How long will it take for one quarter of a given quantity to decay?

b. $\frac{dP}{dt} \sim P$

$\frac{dP}{dt} = kP$

c. INITIAL CONDITION $(t=0)$ $\quad P(0) = P_0$

$\frac{1}{P}\,dP = k\,dt$

$\ln|P| = kt + C$

$|P| = e^{kt+c}$

$|P| = e^{kt} \cdot e^{c}$

$|P| = Ce^{kt}$

$P = Ce^{kt}$

USING $P(0) = P_0$

$P_0 = Ce^{k(0)}$

$P_0 = C$

$\boxed{P = P_0 e^{kt}}$

d. $150 = 100e^{.04t}$

$1.5 = e^{.04t}$

$\ln(1.5) = .04t$

$t \approx 10.137$ yrs

e. $\frac{1}{2} = e^{k(1612)}$

$k \approx -.00043$

$\frac{3}{4} = e^{-.00043t}$

$t \approx 669$ yrs

A DIFFERENT WAY TO LOOK AT IT

$\frac{3}{4} = \left(\frac{1}{2}\right)^{t/1612} \rightarrow$ # OF HALF LIVES

$t \approx 669$ YRS

2. Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

a. Discuss why this intuitively makes sense.
b. Symbolize this proportion and then write an equation relating the values in this proportion.
c. Derive an equation relating the temperature of the object in terms of length of time cooling.
d. You are sitting in Starbucks (constant 73° climate) drinking your Cappuccino which takes 10 minutes to cool from 120 ^oF to 110 ^oF. How many more minutes will it take to cool to 100 ^oF?

$$\frac{dT}{dt} \sim T - T_s$$

$$\frac{dT}{dt} = k(T - T_s)$$

$$\frac{1}{T - T_s}\, dT = k\, dt$$

$$\ln|T - T_s| = kt + c$$

$$|T - T_s| = Ce^{kt}$$

$$T - T_s = Ce^{kt}$$

$$\boxed{T = Ce^{kt} + T_s}$$

C & K ARE BOTH CONSTANTS
SO IS T_s (ASSUMED)

d. $C = 47$ $\quad K \approx -.024$

23.096 MIN.

SO ABOUT 13 MORE MINUTES

3. With Restricted Growth, the rate of change of a quantity y is proportional to the difference between a fixed constant, A, and the current quantity of y present.

a. Symbolize this proportion and then write an equation relating the values in this proportion.

b. Derive a general equation relating the quantity, y, in terms of time, t.

c. Advertisers generally assume that the rate at which people hear about a product for the first time is proportional to the number of people who have not yet heard about it. Suppose that the size of a community is 15,000. At the beginning of an advertising campaign, no one has heard about a product, but after 6 days 1,500 people know about it. How long will it take for 2700 people to have heard of it?

a. $\frac{dy}{dt} \sim (A-y)$

$\frac{dy}{dt} = k(A-y)$

b. $\frac{1}{A-y}\, dy = k\, dt$

$-\ln|A-y| = kt + c$

$\ln|A-y| = kt + c$

$|A-y| = Ce^{kt}$

$A - y = Ce^{kt}$

$-y = Ce^{kt} - A$

$y = Ce^{kt} + A$

c. $A = 15,000$

$y(t) = Ce^{kt} + 15,000$

$y(0) = 0$

$C = -15,000$

$y(t) = -15,000\, e^{kt} + 15000$

$y(6) = 1500$

$k \approx -.01756$

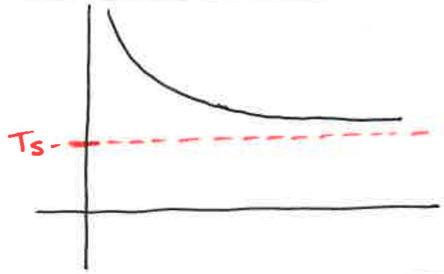
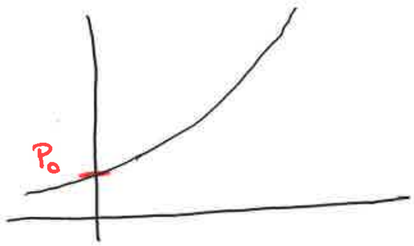
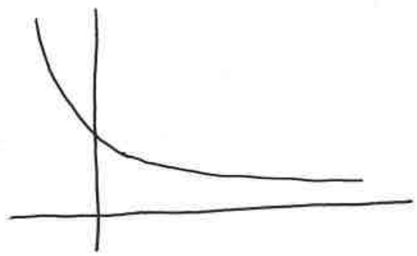
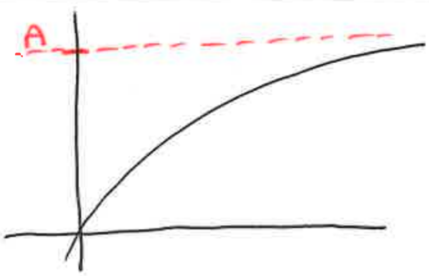
$y(t) = -15000\, e^{-.01756t} + 15,000$

$y(t) = 2700$

$t \approx 11.301$

ABOUT 11 DAYS

4. Organizing Information

Scenario/Equations	Rough Sketch of Graph	Notes
Newton's Law of Cooling – Cappuccino $\frac{dT}{dt} = k(T - T_s)$ $T = Ce^{kt} + T_s$		
Newton's Law of Cooling (considering initial difference) SIMILAR TO ABOVE $\underbrace{T - T_s}_{\text{ENDING } \Delta T} = \underbrace{C}_{\text{INITIAL } \Delta \text{TEMP}} e^{kt}$		
Population Growth in City $\frac{dP}{dt} = kP$ $P = P_0 e^{kt}$		
Cellium-314 Decay $A = Pe^{rt}$ $A = P\left(\frac{1}{2}\right)^{t/(\text{HALF-LIFE})}$		
Cellium-314 Decay (half-life eqtn)	SAME AS ABOVE	
Restricted Growth - Advertising $\frac{dy}{dt} = k(A - y)$ $y = Ce^{kt} + A$		

5. One problem with the "exponential" model for population growth is that it allows for unchecked growth, which is unrealistic given environment considerations. Another model for population growth that incorporates a "cap" on population is called the logistic model for population growth. In this model, ...

The rate of growth of the population is proportional to both the population itself and the difference between the maximum possible population (called the carrying capacity) and the population itself.

a. Express this relationship as a differential equation, using A to represent the carrying capacity, k to represent your constant of proportionality and y to represent the population.
b. Find the general function that satisfies the differential equation. (Explore adjustments to $\frac{dy}{dt}$.)
c. Find the carrying capacity and initial population if the population fits the following model. $P(t) = \frac{108000}{1+17e^{-.3t}}$
d. Because of limited food and space, a squirrel population cannot exceed 1000. It grows at a rate proportional both to the existing population and to the attainable additional population. If there were 100 squirrels 2 years ago, and 1 year ago the population was 400, about how many squirrels are there now?

a. $\frac{dy}{dt} = k \cdot y\left(1 - \frac{y}{A}\right)$

$\frac{A}{A} \cdot \frac{1}{y\left(1-\frac{y}{A}\right)} dy = k\,dt$

$\frac{A}{y(A-y)} dy = k\,dt$

$\frac{A}{y(A-y)} = \frac{B}{y} + \frac{D}{A-y}$

$A = B(A-y) + Dy$

$B = 1 \qquad D = 1$

$\int \frac{1}{y} dy + \int \frac{1}{A-y} dy = \int k\,dt$

$\ln|y| - \ln|A-y| = kt + c$

$\ln\left|\frac{y}{A-y}\right| = kt + c$

$\frac{y}{A-y} = Ce^{kt}$

RECIP.

$\frac{A-y}{y} = Ce^{kt}$

$\frac{A}{y} - 1 = Ce^{kt}$

$\frac{A}{y} = Ce^{kt} + 1$

$\frac{y}{A} = \frac{1}{Ce^{kt}+1}$

$y = \frac{A}{Ce^{kt}+1}$

c. CAP $= 108{,}000 \qquad P_0 = 6000$

d. $y(0) = 100$
$y(1) = 400$
$y(2) = ?$

$100 = \frac{1000}{1+Ce^{k(0)}}$

$C = 9$

$400 = \frac{1000}{1+9e^{k(1)}}$

$k \approx -1.792$

$y(2) = \frac{1000}{1+9e^{-1.792(2)}}$

$= 800$ SQUIRRELS

6. The population of Alaska from 1900 to 2000 can be modeled by the following logistic equation $P(t) = \frac{895598}{1+71.57e^{-0.0516t}}$
Where P is the population t years after 1900.

a. What is the predicted population of Alaska in 2020? 781,218 PEOPLE
b. How fast was the population of Alaska changing in 1920? In 1940? In 1999? $P'(20) \approx 1678$ PPL/YR $P'(40) \approx 4128$ PPL/YR $P'(99) \approx 9742$ PPL/YR
c. When was Alaska growing the fastest, and what was the population at the time?
d. What information does the equation tell us about the population of Alaska in the long run?

c. ABOUT 1982/83

$P'(82.77) \approx 11553$ PPL/YR

7. The growth rate of the population, P, of bears in newly established wildlife preserves is modeled by the differential equation $\frac{dP}{dt} = 0.008P(100 - P)$, where t is measured in years.
a. What is the carrying capacity for bears in this wildlife preserve?
b. What is the bear population when the population is growing the fastest?
c. What is the rate of change of population when it is growing the fastest?

a. 100 BEARS
b. 50 BEARS
c. $P'(50) = 20 \frac{\text{BEARS}}{\text{YR}}$