Convergent and Divergent Sequences

Convergent Sequence – A sequence, S_n, converges to the limit S if $\lim_{n\to\infty} S_n = S$.	Divergent Sequence – a sequence that is not convergent

*For each sequence:

a. List the first 4 terms
b. Identify if it is convergent or divergent. (Explore some ideas together.)
c. If convergent, identify the value of the convergence. (Explore some ideas together.)

1. $a_n = \frac{n^2}{\sqrt{n^3+1}}$ $a_1 = \frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}, \frac{4}{3}, \frac{9}{\sqrt{28}}, \frac{16}{\sqrt{65}}, \ldots$ DIVERGE	2. $a_n = \frac{3n^2}{\sqrt{4n^4+1}}$ $\frac{3}{\sqrt{5}}, \frac{12}{\sqrt{65}}, \frac{27}{\sqrt{325}}, \frac{48}{\sqrt{1025}}, \ldots$ CONVERGE $\frac{3}{2}$
3. $a_n = \cos\left(\frac{2}{n}\right)$ $\cos(2), \cos(1), \cos\left(\frac{2}{3}\right), \cos\left(\frac{1}{2}\right)$ CONVERGE 1	4. $a_n = \sin(\sqrt{n})$ $\sin(1), \sin(\sqrt{2}), \sin(\sqrt{3}), \sin(2)$ DIVERGE
5. $a_n = \frac{x^n}{n!}$ where x is any real number $x^1, \frac{x^2}{2}, \frac{x^3}{6}, \frac{x^4}{24}$ CONVERGE 0	6. $a_n = \frac{(-1)^n}{\cos(1/n)}$ $\frac{-1}{\cos(1)}, \frac{1}{\cos(\frac{1}{2})}, \frac{-1}{\cos(1/3)}, \frac{1}{\cos(1/4)}$ DIVERGE
7. $a_n = 1 + \left(-\frac{9}{10}\right)^n$ $\frac{1}{10}, 1\frac{81}{100}, 1 - \frac{9^3}{1000}, 1 + \frac{9^4}{10000}$ CONVERGE 1	8. $a_n = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}$ $1, \frac{3}{2}, \frac{5}{6}, \frac{7}{24}, \ldots$ DIVERGE

Introduction to Convergent/Divergent Geometric Series

Let's begin with Geometric Sequence.

1. Given: $7, \frac{7}{2}, \frac{7}{4}, \frac{7}{8}, \ldots$ a. Write an explicit formula for this sequence. b. Is this sequence convergent or divergent. a. $y = 7\left(\frac{1}{2}\right)^{x-1}$ OR $a_n = 14\left(\frac{1}{2}\right)^n$ b. SEQUENCE IS CONVERGENT	2. Given: $5, \frac{25}{3}, \frac{125}{9}, \frac{625}{27}, \ldots$ a. Write an explicit formula for this sequence. b. Is this sequence convergent or divergent. a. $y = 5\left(\frac{5}{3}\right)^{x-1}$ b. SEQUENCE IS DIVERGENT

3. What should you be looking for to determine whether a geometric sequence is convergent or divergent? Why does this make sense? GEOMETRIC SERIES TEST FOR CONVERGENCE OR DIVERGENCE

IF $|r| < 1$, SERIES CONVERGES

IF $|r| > 1$, SERIES DIVERGES

4. We recently derived the formula for the sum of a geometric series to be $S = \frac{a(r^n - 1)}{r - 1}$ or $\frac{a(1 - r^n)}{1 - r}$	5. We can look at this geometric summation as an infinite series where $n = \infty$. $\sum_{n=1}^{\infty} a \cdot r^{n-1}$
6. What happens to this formula for a geometric series when $n \to \infty$ and $\|r\| > 1$? $S = \frac{a(r^n - 1)}{r - 1}$ or $\frac{a(1 - r^n)}{1 - r}$ IF $\|r\| > 1$ $\lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \infty$	7. What happens to this formula for a geometric series when $n \to \infty$ and $\|r\| < 1$? $S = \frac{a(r^n - 1)}{r - 1}$ or $\frac{a(1 - r^n)}{1 - r}$ IF $\|r\| < 1$ $\lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a_1}{1 - r}$

8. Analysis of the Infinite Geometric Series

$$\sum_{n=1}^{\infty} 4 \cdot \left(\frac{-3}{4}\right)^{n-1}$$

a. What do you think the result of the infinite sum will be?
b. What are the first six terms in the sequence?
c. What are the first six summation terms of the series?
d. What will the graph of the first 6 summation terms looks like?

a.

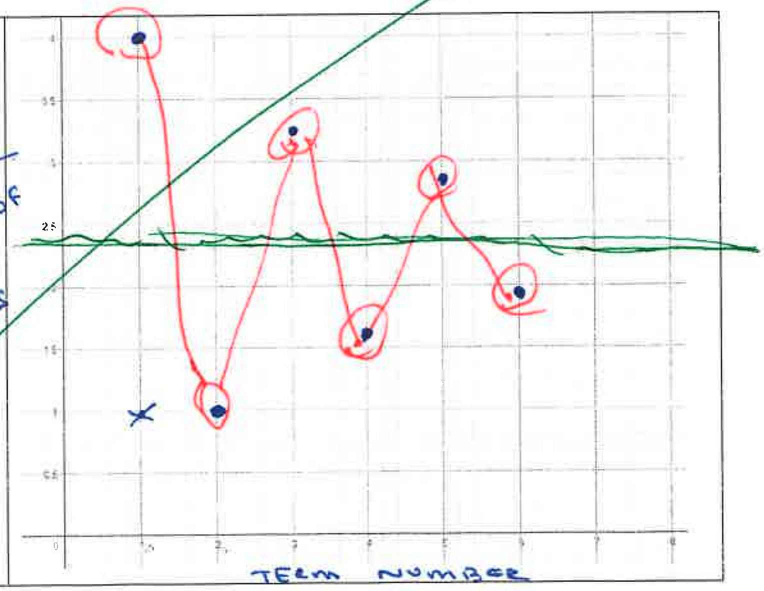
b. $4, -3, \frac{9}{4}, -\frac{27}{16}, \frac{81}{64}, -\frac{243}{256}$

c.

SUM OF FIRST n TERMS

a. $\frac{4}{1 - \left(\frac{-3}{4}\right)} = \frac{16}{7}$

$= 4 \div \frac{7}{4}$



The nth Term Divergence Test

Notes: Use $\sum_{n=1}^{\infty} \frac{5n^2-1}{2n^2}$ ONLY TO TEST FOR DIVERGENCE

SPSE a_n IS AN ALGEBRAIC EXPRESSION FOR A SERIES

IF $\lim_{n\to\infty} a_n \neq 0$, THE SERIES DIVERGES

IF $\lim_{n\to\infty} a_n = 0$, NO CONCLUSION

1.

$$\sum_{n=0}^{\infty} \frac{4n^2-n^3}{7-3n^3}$$

$$\lim_{n\to\infty} \frac{4n^2-n^3}{7-3n^3} = \lim_{n\to\infty} \frac{-n^3}{-3n^3} = \frac{1}{3}$$

SERIES DIVERGES

2.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

NO CONCLUSION

3.

$$\sum_{n=1}^{\infty} \frac{n+2}{2n^2+7}$$

$$\lim_{n\to\infty} \frac{n}{2n^2} = 0$$

NO CONCLUSION

4.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

NO CONCLUSION

5.

$$\sum_{n=1}^{\infty} ne^n$$

$$\lim_{n\to\infty} n \cdot e^n = \infty$$

DIVERGES

6.

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\lim_{n\to\infty} \frac{n^n}{n!} = \infty$$

DIVERGES

Practice WS with Series (Converge or Diverge)

The Ratio Test – Method to Prove Convergence or Divergence

Notes: Use $\sum_{n=0}^{\infty} \frac{4^n}{n!}$

$1 + 4 + 8 + \frac{32}{3} + \ldots$

$a_n = \frac{4^n}{n!} \qquad a_{n+1} = \frac{4^{n+1}}{(n+1)!}$

RATIO TEST

$\lim_{n\to\infty} \left| \frac{4^{n+1}/(n+1)!}{4^n/n!} \right|$

RATIO TEST

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

IF $L < 1$, SERIES CONVERGE

IF $L > 1$, SERIES DIVERGES

IF $L = 1$, INCONCLUSIVE

1. RATIO TEST $\sum_{n=0}^{\infty} \frac{n!}{5^n}$

$\lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{5^n}{5^{n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)\cdot n!}{n!} \cdot \frac{5^n}{5^n \cdot 5} \right|$

$= \lim_{n\to\infty} \left| \frac{n+1}{5} \right| = \infty$

SERIES DIVERGES BECAUSE $L > 1$.

2. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ where x is any real number

RATIO TEST

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right|$

$= \lim_{n\to\infty} \left| \frac{x^n \cdot x \cdot n!}{x^n \cdot (n+1) \cdot n!} \right|$

$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$

SINCE $L < 1$, SERIES CONVERGES,

3. RATIO TEST $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$

$\lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{(2n-1)!}{(2(n+1)-1)!} \right|$

$= \lim_{n\to\infty} \left| \frac{(n^2+2n+1)}{n^2} \frac{(2n-1)!}{(2n+1)!} \right|$

$= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right| \cdot \lim_{n\to\infty} \left| \frac{(2n-1)!}{(2n+1)(2n)\cdot(2n-1)!} \right|$

$= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right| \cdot \lim_{n\to\infty} \left| \frac{1}{(2n+1)(2n)} \right|$

$= 1 \cdot 0 = 0$

SINCE $L < 1$, SERIES CONVERGES

4. RATIO TEST $\sum_{n=1}^{\infty} \frac{9^n}{(-2)^{n+1}(n)}$

$\lim_{n\to\infty} \left| \frac{9^{n+1}}{9^n} \frac{(-2)^{n+1}}{(-2)^{n+2}} \cdot \frac{(n)}{(n+1)} \right|$

$= \lim_{n\to\infty} \left| \frac{9}{(-2)} \frac{(n)}{(n+1)} \right|$

$= \lim_{n\to\infty} \left| \frac{9}{-2} \right| \cdot \lim_{n\to\infty} \left| \frac{n}{n+1} \right|$

$= \frac{9}{2} \cdot 1 = \frac{9}{2}$

SINCE $L > 1$, SERIES DIVERGES.

5. $$\sum_{n=1}^{\infty} \frac{n+2}{2n+7}$$ $\lim_{n\to\infty} \left\lvert \frac{n+3}{2n+9} \cdot \frac{2n+7}{n+2} \right\rvert$ $= \lim_{n\to\infty} \left\lvert \frac{2n^2}{2n^2} \right\rvert = 1$ NO CONCLUSION	6. $$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$ $\lim_{n\to\infty} \left\lvert \frac{(-10)^{n+1}}{4^{2n+3}(n+2)} \cdot \frac{4^{2n+1}(n+1)}{(-10)^n} \right\rvert$ $= \lim_{n\to\infty} \left\lvert \frac{(-10)(-10)^n}{4^{2n+1}\cdot 4^2 (n+2)} \cdot \frac{4^{2n+1}(n+1)}{(-10)^n} \right\rvert$ $= \lim_{n\to\infty} \left\lvert \frac{(-10)}{16} \frac{(n+1)}{(n+2)} \right\rvert = \frac{5}{8}$ SINCE $L<1$, SERIES CONVERGES
7. $$\sum_{n=1}^{\infty} \frac{1}{3n}$$ $\lim_{n\to\infty} \left\lvert \frac{1}{3n+3} \cdot \frac{3n}{1} \right\rvert = 1$ NO CONCLUSION	8. $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ $\lim_{n\to\infty} \left\lvert \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} \right\rvert = 1$ NO CONCLUSION
9. $$\sum_{n=1}^{\infty} ne^n$$ $\lim_{n\to\infty} n\cdot e^n = \infty$ $\lim_{n\to\infty} \left\lvert \frac{(n+1)\, e^{n+1}}{n\, e^n} \right\rvert = \frac{e^n \cdot e}{e^n} = e$ SINCE $L>1$, SERIES DIVERGES	10. $$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$ $\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$ $\lim_{n\to\infty} \frac{(n+1)^n (n+1)}{(n+1)\; n!} \cdot \frac{n!}{n^n}$ $= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = e$ SINCE $L>1$, SERIES DIVERGES $\left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n = e$

$\cancel{\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^p}$

p-Series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a constant (a power) and n is the incremental value in the series

Explore: Write out the first several terms in each series and write whether you believe the series converges or diverges.

$\sum_{n=1}^{\infty} \frac{1}{n^{-3}} = 1 + 8 + 27 + 64 + \dots = \sum_{n=1}^{\infty} n^3$

NTH TERM DIVERGENCE TEST

$L = \lim_{n\to\infty} n^3 = \infty$

SINCE $L \neq 0$, SERIES ~~CONVERGES~~ DIVERGES

$\sum_{n=1}^{\infty} \frac{1}{n^{-2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{-\frac{3}{2}}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + \dots$

$\sum_{n=1}^{\infty} \frac{1}{n^{-\frac{1}{2}}} = \sum_{n=1}^{\infty} \sqrt{n} = 1 + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \dots$

$\sum_{n=1}^{\infty} \frac{1}{n^0} = 1 + 1 + 1 + 1 + 1 + 1 + 1$ DIVERGES

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \dots$

DIV $\sum_{n=1}^{\infty} \frac{1}{n^1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} = 1 + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{27}} + \frac{1}{8} + \dots$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$

$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots$ CONVERGE

Notes on p-Series:

IN A P-SERIES:

IF $p \leq 1$, SERIES DIVERGES

IF $p > 1$, SERIES CONVERGES

Improper Integrals – Integrals that cannot be computed using a normal Riemann Integral

What causes this?

1. The interval of integration is infinite
2. The integrand is discontinuous in the interval

We will use limits to assist us in calculating these definite integrals.

1. $\int_1^\infty \frac{1}{x}dx$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} \Big[\ln|b| - \ln(1)\Big]$$

$$= \infty$$

2. $\int_1^\infty \frac{1}{x^2}dx$

$$= \lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty} \Big[-x^{-1}\Big]_1^b$$

$$= \lim_{b\to\infty} \Big[\frac{-1}{b} - \frac{-1}{1}\Big] = 1$$

3. $\int_0^1 \frac{1}{x}dx$

$$\lim_{b\to0^+} \int_b^1 \frac{1}{x}\,dx$$

$$= \lim_{b\to0^+} \Big[\ln|x|\Big]_b^1$$

$$= \lim_{b\to0^+} \Big[\ln|1| - \ln|b|\Big]$$

$$= \infty$$

4. $\int_0^1 \frac{1}{\sqrt{x}}dx$

$$\lim_{b\to0^+} \int_b^1 x^{-1/2}\,dx$$

$$= \lim_{b\to0^+} \Big[2x^{1/2}\Big]_b^1$$

$$= \lim_{b\to0^+} \Big[2 - 2\sqrt{b}\Big]$$

$$= 2$$

5. $\int_0^\infty e^{-x}dx$

$$\lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} -e^{-x}\Big]_0^b$$

$$= \lim_{b\to\infty} \Big[-e^{-b} + e^{0}\Big]$$

$$= 1$$

6. $\int_{-\infty}^\infty xe^{-x^2}dx$

$$\lim_{a\to-\infty} \int_a^0 x e^{-x^2}\,dx + \lim_{b\to\infty} \int_0^b x e^{-x^2}\,dx$$

$$= \lim_{a\to-\infty} -\frac{1}{2}e^{-x^2}\Big]_a^0 + \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big]_0^b$$

$$= \lim_{a\to-\infty} \Big[-\frac{1}{2} + \frac{1}{2}e^{-a^2}\Big] + \lim_{b\to\infty} \Big[-\frac{1}{2}e^{-b^2} + \frac{1}{2}\Big]$$

$$\Big[-\frac{1}{2}\Big] + \Big[\frac{1}{2}\Big] = 0$$

$$u = -x^2 \qquad -\frac{1}{2}\int e^u\,du$$

$$du = -2x\,dx$$

$u = \ln x \quad du = \frac{1}{x}dx$

7. $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

$\lim_{b\to\infty} \int_e^b u^{-2}\,du$

$= \lim_{b\to\infty} -u^{-1}\Big]_e^b$

$= \lim_{b\to\infty} \frac{-1}{\ln x}\Big]_e^b$

$= \lim_{b\to\infty} \left[\frac{-1}{\ln b} + \frac{1}{\ln e}\right]$

$= 1$

8. $\int_{-\infty}^\infty x^5\,dx$

$\lim_{b\to-\infty} \int_b^0 x^5\,dx + \lim_{a\to\infty} \int_0^a x^5\,dx$

$= \lim_{b\to-\infty} \frac{1}{6}x^6\Big]_b^0 + \lim_{a\to\infty} \left[\frac{1}{6}x^6\right]_0^a$

$= \lim_{b\to-\infty} -\frac{1}{6}b^6 + \lim_{a\to\infty} \frac{1}{6}a^6$

$= 0$

9. $\int_1^\infty \ln x\,dx$

$\lim_{b\to\infty} \int_1^b \ln(x)\,dx$

$= \lim_{b\to\infty} \left[x\ln(x) - x\right]_1^b$

$= \lim_{b\to\infty} \left[b\ln(b) - b\right] - [-1]$

$= \lim_{b\to\infty} \left[b(\ln(b) - 1)\right] + 1$

$= \infty$

10. $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

$\lim_{b\to 2} \int_0^b \frac{1}{\sqrt{4-x^2}}\,dx$

$= \frac{1}{2} \lim_{b\to 2} \int_0^b \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,dx \qquad u = \frac{x}{2} \quad du = \frac{1}{2}dx$

$= \lim_{b\to 2} \arcsin(\frac{x}{2})\Big]_0^b$

$= \lim_{b\to 2} \arcsin(\frac{b}{2}) - \arcsin 0$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$

11. $\int_0^4 \frac{x}{\sqrt{16-x^2}}\,dx \qquad u = 16 - x^2 \quad du = -2x\,dx$

$\lim_{b\to 4^-} -\frac{1}{2}\int_0^b \frac{-2x}{\sqrt{16-x^2}}\,dx$

$= \lim_{b\to 4^-} -\frac{1}{2}\int u^{-1/2}\,du = \lim_{b\to 4^-} -\frac{1}{2}\left[2u^{1/2}\right]$

$= -1 \cdot \lim_{b\to 4^-} \sqrt{16-x^2}\Big]_0^4$

$= -1\left[0 - 4\right] = 4$

12. $\int_{-\infty}^\infty \frac{x}{(1+x^2)^2}\,dx \qquad u = 1+x^2 \quad du = 2x\,dx$

$= \lim_{a\to-\infty} \int_a^0 \frac{x}{(1+x^2)^2}dx + \lim_{b\to\infty} \int_0^b \frac{x}{(1+x^2)^2}\,dx$

$= \lim_{a\to-\infty} \frac{1}{2}\int u^{-2}\,du + \lim_{b\to\infty} \frac{1}{2}\int u^{-2}\,du$

$= \lim_{a\to-\infty} -\frac{1}{2}u^{-1}\Big] + \lim_{b\to\infty} -\frac{1}{2}\left[u^{-1}\right]$

$= \lim_{a\to-\infty} \frac{-1}{2(1+x^2)}\Big]_a^0 + \lim_{b\to\infty} \left[\frac{-1}{2(1+x^2)}\right]_0^b$

$= \left[-\frac{1}{2} - 0\right] + \left[0 - (-\frac{1}{2})\right]$

$= 0$

Integral Test

Given $\sum a_n$ where $a_n = f(x)$ is a continuous, positive decreasing function then,

1. If $\int_k^\infty f(x)dx$ is convergent so is $\sum_{n=k}^\infty a_n$.

2. If $\int_k^\infty f(x)dx$ is divergent so is $\sum_{n=k}^\infty a_n$.

*Show that each series is convergent or divergent using the Integral Test.

1. $\sum_{n=1}^\infty \frac{1}{n}$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to\infty}\Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} \ln|b| - \ln|1| = \infty$$

SINCE INTEGRAL IS DIVERGENT, SERIES IS ALSO DIVERGENT

2. $\sum_{n=2}^\infty \frac{1}{n\ln(n)}$

LET $u = \ln(\cancel{x})$, $du = \frac{1}{\cancel{x}}dx$

$$\lim_{b\to\infty} \int_2^b \frac{1}{u}\,du$$

$$= \lim_{b\to\infty} \ln|\ln(x)|\Big]_2^b$$

$$= \infty$$

SINCE INTEGRAL IS DIVERGENT, SO IS THE SERIES

3. $\sum_{n=2}^\infty \frac{n}{(n+2)^{\frac{5}{2}}}$

LET $u = \cancel{x}+2$

$x = u-2$

$du = dx$

$$\int_2^\infty \frac{u-2}{u^{5/2}}\,du = \int_2^\infty u^{-3/2}\,du - 2\int_2^\infty u^{-5/2}\,du$$

$$= \lim_{b\to\infty} -2(x+2)^{-1/2} + \frac{4}{3}(x+2)^{-3/2}\Big]_2^b$$

$$= (0) - (-1) + (0) - \left(\frac{1}{6}\right)$$

$$= \frac{5}{6}$$

SINCE INTEGRAL IS CONVERGENT SO IS THE SERIES

4. $\sum_{n=2}^\infty \frac{\ln n}{n^2}$

$$\int_2^\infty \frac{\ln(x)}{x^2}\,dx$$

$u = \ln(x)$ $\quad dv = x^{-2}dx$

$du = \frac{1}{x}dx$ $\quad v = -\frac{1}{x}$

$$= -\frac{\ln(x)}{x} + \int \frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} -\frac{\ln(x)}{x} - \frac{1}{x}\Big]_2^b$$

$$= \lim_{b\to\infty}\left[\left(-\frac{\ln(b)}{b} - \frac{1}{b}\right) - \left(-\frac{\ln(2)}{2} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\ln(2)+1\right]$$

CONVERGES

5. $\sum_{n=1}^\infty \frac{n}{(n^2+2)^{\frac{3}{2}}}$

$u = x^2+2$

$du = 2x\,dx$

$$\frac{1}{2}\int_1^\infty \frac{2x}{(x^2+2)^{3/2}}\,dx$$

$$\frac{1}{2}\int u^{-3/2}\,du$$

$$\frac{1}{2}\cdot -2u^{-1/2}\Big] = \frac{-1}{\sqrt{u}}$$

$$= \lim_{b\to\infty}\left[\frac{-1}{\sqrt{x^2+2}}\right]_1^b = \lim_{b\to\infty}\left[(0) - \left(\frac{-1}{\sqrt{3}}\right)\right]$$

$$\frac{1}{\sqrt{3}}$$

CONVERGES

6. $\sum_{n=1}^\infty n^2 e^{-n^3}$

$u = -x^3$

$du = -3x^2\,dx$

$$-\frac{1}{3}\int_1^\infty -3x^2 \cdot e^{-x^3}\,dx$$

$$= -\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^u\Big]$$

$$= -\frac{1}{3}\lim_{b\to\infty}\left[e^{-x^3}\right]^b$$

$$= -\frac{1}{3}\lim_{b\to\infty}\left[\left(e^{-b^3}\right) - \left(e^{-1}\right)\right]$$

$$= \frac{1}{3e}$$

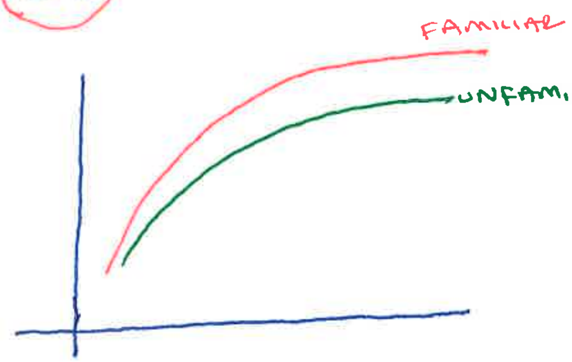
CONVERGES

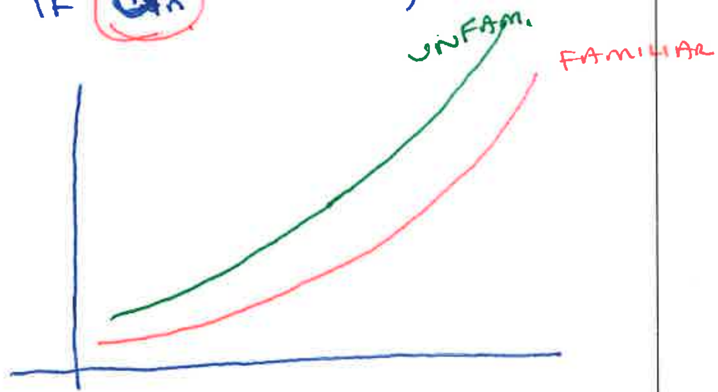
DIRECT

Comparison Test: You are comparing an Unfamiliar Series with a Familiar Series to determine Convergence or Divergence.

Familiar Series are oftentimes Geometric Series or p-Series.

Convergence	Divergence
~~$a_n, b_n \geq 0$~~	~~$a_n, b_n \geq 0$~~
$0 \leq a_n \leq b_n$ $b_n \geq a_n > 0$	$0 \leq a_n \leq b_n$ $b_n \geq a_n > 0$
FAMILIAR; LARGER	FAMILIAR; SMALLER
IF b_n CONVERGES, SO DOES a_n	IF a_n DIVERGES, SO DOES b_n





1. Does $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converge or diverge?

$= \frac{1}{3} + \frac{1}{6} + \frac{1}{11} + \cdots$

COMPARISON TEST

COMPARE W/ CONVERGING GEO. SERIES

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

B.C. $\frac{1}{2^n} \geq \frac{1}{2^n+n}$ $\forall n$ & $\sum_{n=1}^{\infty} \frac{1}{2^n}$,

SO WILL $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ ALSO CONVERGE.

2. Does $\sum_{n=1}^{\infty} \frac{n+6^n}{5^n}$ converge or diverge?

$= \frac{7}{5} + \frac{38}{25} + \frac{219}{125} + \cdots$

COMPARISON TEST

COMPARE W/ DIVERGING GEO. SERIES

$\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n = \sum_{n=1}^{\infty} \frac{6^n}{5^n} = \frac{6}{5} + \frac{36}{25} + \frac{216}{125}$

BECAUSE OUR SERIES DIVERGES AND IS LESS THAN $\sum_{n=1}^{\infty} \frac{n+6^n}{5^n}$, THIS SERIES ALSO DIVERGES.

3. Does $\sum_{n=1}^{\infty} \frac{7}{n(n+1)}$ converge or diverge?

$= \frac{7}{2} + \frac{7}{6} + \frac{7}{12} + \frac{7}{20} + \cdots$

COMPARISON TEST

COMPARE WITH CONVERGING P-SERIES

$7 \sum_{n=1}^{\infty} \frac{1}{n^2} = 7 + \frac{7}{4} + \frac{7}{9} + \frac{7}{16} + \cdots$

4. Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots$

COMPARISON TEST   COMPARE WITH DIVERGE

$\cancel{\frac{1}{2} + \frac{1}{2}} + \cancel{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}} + \cancel{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \cdots$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \cdots$

BECAUSE OUR SERIES DIVERGES & IS LESS THAN $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ WILL ALSO DIVERGE.

Intuition - Converging or Diverging? $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Can you use the Comparison Test to Prove your intuition? Why or why not?

GUT SAYS CONVERGENT

$$\sum_{n=1}^{\infty} \frac{1}{2^n-1} = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots$$

COMPARE W/ GEOMETRIC SERIES (CONVERGENT)

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

THIS SERIES MUST BE LARGER THAN $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

TO PROVE CONVERGENCE ... BUT IT IS NOT.

The Limit Comparison Test

TWO SERIES $\sum a_n$, $\sum b_n$ WHERE $a_n \geq 0$, $b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

IF L IS POSITIVE & FINITE BOTH SERIES EITHER CONVERGE OR DIVERGE.

Use either the comparison test or limit comparison test to prove whether each series converges or diverges. In some cases either test is appropriate. In other cases, only one test will work.

1. $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

BOTH SERIES > 0

COMPARE $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^n-1}} = \lim_{n\to\infty} \frac{2^n-1}{2^n}$$

$$= \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$$

LIMIT IS POSITIVE & FINITE

BOTH SERIES ARE CONVERGENT.

2. $\sum_{n=1}^{\infty} \frac{n+e^n}{\pi^n}$

COMPARE W/ CONVERGENT $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ OR $\frac{e^n}{\pi^n}$

$$\lim_{n\to\infty} \frac{\frac{n+e^n}{\pi^n}}{\frac{e^n}{\pi^n}} = \lim_{n\to\infty} \frac{(n+e^n)\,\pi^n}{\pi^n \quad e^n}$$

$$= \lim_{n\to\infty} \frac{n+e^n}{e^n} = \lim_{n\to\infty} \frac{n}{e^n} + 1$$

$$= 1$$

BOTH SERIES ARE CONVERGENT.

3. $\sum_{n=1}^{\infty} \frac{1}{4^n+6} = \frac{1}{10} + \frac{1}{22} + \cdots$

COMPARE W/ $\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{4} + \frac{1}{16} + \cdots$ CONVERGENT GEOMETRIC

DIRECT COMPARISON

SINCE $\sum_{n=1}^{\infty} \frac{1}{4^n} \geq \sum_{n=1}^{\infty} \frac{1}{4^n+6}$,

$\sum_{n=1}^{\infty} \frac{1}{4^n+6}$ IS ALSO CONVERGENT.

4. $\sum_{n=1}^{\infty} \frac{n^2}{1+n} = \frac{1}{2} + 2 + \frac{9}{4} + \frac{16}{5}$

$\frac{n^2}{n}$

COMPARE W/ DIVERGENT $\sum_{n=1}^{\infty} \frac{1}{n^{-1}}$ AKA n

p-SERIES

LIMIT COMPARISON

$\lim_{n\to\infty} \frac{n}{\frac{n^2}{1+n}} = \lim_{n\to\infty} \frac{n(1+n)}{n^2}$

$= \lim_{n\to\infty} \frac{n+n^2}{n^2} = \lim_{n\to\infty} \frac{n}{n^2} + \frac{n^2}{n^2}$

$= \lim_{n\to\infty} \frac{1}{n} + 1 = 1$

POSITIVE & FINITE, SO BOTH ARE DIVERGENT

5. $\sum_{n=1}^{\infty} \frac{3^n}{5^n+n} = \frac{1}{2} + \frac{9}{27} + \frac{27}{128} + \cdots$

COMPARE WITH CONVERGING $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ GEOMETRIC

DIRECT COMPARISON

SINCE $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n \geq \sum_{n=1}^{\infty} \frac{3^n}{5^n+n}$

$\sum_{n=1}^{\infty} \frac{3^n}{5^n+n}$ IS ALSO CONVERGENT

6. $\sum_{n=4}^{\infty} \frac{1}{n^2-8} = \frac{1}{8} + \frac{1}{17} + \frac{1}{28}$

COMPARE WITH CONVERGENT p-SERIES

$\sum_{n=4}^{\infty} \frac{1}{n^2} = \frac{1}{16} + \frac{1}{25}$

LIMIT COMPARISON

$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2-8}} = \lim_{n\to\infty} \frac{n^2-8}{n^2}$

$= \lim_{n\to\infty} 1 - \frac{8}{n^2} = 1$

POSITIVE & FINITE, SO BOTH ARE CONVERGENT.

7. $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} = \frac{1}{2} + \frac{1}{2+\sqrt{2}} + \frac{1}{3+\sqrt{3}} + \cdots$

COMPARE W/ DIVERGENT

$\sum_{n=1}^{\infty} \frac{1}{n+n} = \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$

SINCE $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{2n}$,

$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ IS ALSO DIVERGENT.

8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4+1}} = \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{17}} + \frac{1}{\sqrt[3]{82}} + \cdots$

COMPARE W/ CONVERGENT p-SERIES.

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} = 1 + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{81}} + \cdots$

BECAUSE $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4+1}}$,

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4+1}}$ IS ALSO CONVERGENT.

4. Review! Decide whether each of the following is a (constant multiple of a) p-series, geometric series, or neither. For those which are p-series or geometric series, determine whether they converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^{n-1}}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(3n)^5}$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^{n-1}}$

e) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{2}{n}}$ f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{2n}}$

Also determine the value of the constant multiple, a.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

P-SERIES $p > 1$

CONVERGING

$a = 1$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^n \cdot 4^{-1}} = 12 \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$

GEOMETRIC SERIES $|r| < 1$

CONVERGENT

$a = 12$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^5 n^5} = \frac{1}{3^5} \sum_{n=1}^{\infty} \frac{1}{n^4}$

P-SERIES $p > 1$

CONVERGING

$a = \frac{1}{3^5}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n} \cdot 2}{5^n \cdot 5^{-1}} = \sum_{n=1}^{\infty} \frac{4^n \cdot 10}{5^n}$

$\displaystyle = 10 \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$

GEOMETRIC SERIES

CONVERGENT $|r| < 1$

$a = 10$

e. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2}}{\sqrt{n}}$

$\displaystyle = \sqrt{2} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

P-SERIES $p \le 1$

DIVERGENT

$a = \sqrt{2}$

f. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

ALTERNATING HARMONIC SERIES

$\displaystyle\lim_{n\to\infty} a_n = 0$ a_n IS DECREASING

CONVERGENT SERIES

$a = 1$

g. $\displaystyle\sum_{n=1}^{\infty} \frac{(2 \cdot -1)^n}{(e^2)^n}$

$\displaystyle = \sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{e^2}\right)^n$

GEOMETRIC SERIES $|r| < 1$

CONVERGENT

$a = 1$

The Alternating Series Test

Given Alternating Series $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$ where $a_n \geq 0$ for all n. Then if,

1. $\lim\limits_{n\to\infty} a_n = 0$ and
2. a_n is a decreasing sequence

the series is convergent....else, inconclusive result.

*Determine if each of the following in convergent or if the Alternating Series Test is inconclusive.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $= \sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ..$

$\lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\frac{1}{n}$ IS A DECREASING SEQUENCE

$\therefore \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ CONVERGES

2. $\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{2n}}$ $= \sum\limits_{n=1}^{\infty} \frac{(-1)^n (2)^n}{(e^2)^n}$

$= \sum\limits_{n=1}^{\infty} (-1)^n \left(\frac{2}{e^2}\right)^n$

$\lim\limits_{n\to\infty} \left(\frac{2}{e^2}\right)^n = 0$ $|r| < 1$

DECREASING SEQ.

$\therefore \sum\limits_{n=1}^{\infty} \frac{(-2)^n}{e^{2n}}$ CONVERGES

3. $\sum_{n=1}^{\infty}(-1)^n \frac{4n^2}{n^3+9}$

$\lim\limits_{n\to\infty} \frac{4n^2}{n^3+9} = 0$

$\frac{4}{10}, \frac{16}{17}, \frac{36}{36}, \frac{64}{73}$

INCREASING, THEN DECREASING

DECREASING FOR $n > 2$.

CONVERGES

4. $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n}$ $= \sum\limits_{n=1}^{\infty} \frac{1}{n}$

DIVERGENT

5. $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$ $= \sum\limits_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

$\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

SEQUENCE DECREASES

$\sum\limits_{n=2}^{\infty} \cos(n\pi)$ CONVERGES

6. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$ $= \sum\limits_{n=1}^{\infty} \frac{(-1)^n (2)^n}{(3)^n} = \sum\limits_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n$

$\lim\limits_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

SEQUENCE IS DECREASING

$\sum\limits_{n=1}^{\infty} \frac{(-2)^n}{3^n}$ CONVERGES

Given: $f(x) = \cos(x)$

a. Determine a **line** $P(x)$ that closely matches the characteristics of $f(x)$ at the point $(0, 1)$.

$P(x) =$ 1

b. Use Geogebra to visualize how $f(x)$ and $P(x)$ interact.

c. Fill in the following table (rounding to 3 decimal places when necessary).

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	.995	1	.005
.5	.87758	1	.12242
1	.5403	1	.4597

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$f'(0) = 0$

$y = 1$

$P(x) = 1 + 0x$

$p(x) = ax + b$

$p'(x) = a$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

Given: $f(x) = \cos(x)$

a. Determine a **parabola** $P(x)$ that closely matches the characteristics of $f(x)$ at the point $(0, 1)$.

$P(x) =$ $-\frac{1}{2}x^2 + 1$

$p(x) = 1 + 0x - \frac{1}{2}x^2$

b. Use Geogebra to visualize how $f(x)$ and $P(x)$ interact.

c. Fill in the following table (rounding to 3 decimal places when necessary).

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	.995	.995	0
.5	.87758	.875	.00258
1	.5403	.5	.0403

$p(x) = ax^2 + bx + c$

$p(0) = 1$

$p'(x) = 2ax + b$

$p'(0) = 0$

$p''(x) = 2a$

$p''(0) = -1$

$2a = -1$ $(0,1)$

$a = -\frac{1}{2}$

$2ax + b = 0$

$2(-\frac{1}{2})(0) + b = 0$

$b = 0$

$ax^2 + bx + c = 1$

$-\frac{1}{2}(0)^2 + 0 + c = 1$

$c = 1$

$f(x) = \cos(x)$

$f(0) = 1$

$f'(x) = -\sin(x)$

$f'(0) = 0$

$f''(x) = -\cos(x)$

$f''(0) = -1$

Given: $f(x) = \cos(x)$

Determine a **fourth degree polynomial**, $P(x)$, that closely matches the characteristics of $f(x)$ when centered at $x = 0$.

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$P'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$P''(x) = 12ax^2 + 6bx + 2c$

$P'''(x) = 24ax + 6b$

$P^{(4)}(x) = 24a$

$24a = 1$
$a = \frac{1}{24}$

$6b = 0 \quad b = 0$

$2c = -1$
$c = -\frac{1}{2}$

$d = 0$

$e = 1$

$f(x) = \cos(x)$
$f(0) = 1$

$f'(x) = -\sin(x)$
$f'(0) = 0$

$f''(x) = -\cos(x)$
$f''(0) = -1$

$f'''(x) = \sin(x)$
$f'''(0) = 0$

$f^{(4)}(x) = \cos(x)$
$f^{(4)}(0) = 1$

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	.995	.995	0
.5	.87758	.8776	.000022
1	.5403	.54167	.00136

$P(x) = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{24}x^4 \qquad P(x) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$

Given: $g(x) = e^x$

Determine a **fourth degree polynomial**, $P(x)$, that closely matches the characteristics of $f(x)$ when centered at $x = 0$.

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$P'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$P''(x) = 12ax^2 + 6bx + 2c$

$P'''(x) = 24ax + 6b$

$P^{(4)}(x) = 24a$

$24a = 1 \quad a = \frac{1}{24}$

$6b = 1 \quad b = \frac{1}{6}$

$2c = 1 \quad c = \frac{1}{2}$

$d = 1$

$e = 1$

$P(x) = \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0	1	1	0
.1	1.1052	1.1052	0
.5	1.6487	1.6484	.00028
1	2.7183	2.7083	.00995

Taylor Series as an infinite sum.

Write a Taylor/**Maclaurin (centered at $x = 0$)** Series polynomial function with at least 4 terms for each of the following functions. Then write the polynomial function as an infinite series. Use a separate sheet of paper for work.

1. $f(x) = \cos(x)$	$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
2. $f(x) = e^x$	$P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
3. $f(x) = \ln(1+x)$	$P(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$
4. $f(x) = \sin(x)$	$P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
5. $f(x) = \frac{1}{1-x}$	$P(x) = 1 + x + x^2 + x^3 + \ldots$	$\sum_{n=0}^{\infty} x^n$
6. $f(x) = tan^{-1}x$	$P(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$$\sum_{n=0}^{\infty} r^n = \frac{a_1}{1-r}$$

$$y = x^n$$

$$\frac{1}{2n+1} \cdot x^{2n+1}$$

Investigating the Taylor/MacLaurin Series for $f(x) = tan^{-1}x$ centered at $x = 0$.

$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$g(x) = tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
$f'(x) = -1(1-x)^{-2} = \frac{-1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$	$g'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$