Polar Equations – BC Calculus   1. Discuss Polar vs. Cartesian Coordinates $y = f(\theta)$ vs. $y = f(x)$

2.

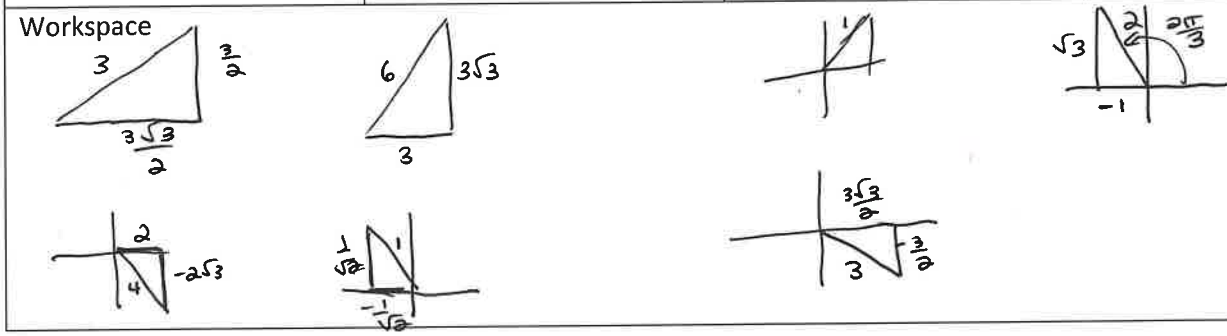
Converting Rectangular to Polar

x	y	θ	r
$\frac{3\sqrt{3}}{2}$	$\frac{3}{2}$	$\frac{\pi}{6}$ $30°$	3
3	$3\sqrt{3}$	$\frac{\pi}{3}$	6
2	$-2\sqrt{3}$	$-\pi/3$	4
$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{3\pi}{4}$	1

Converting Polar to Rectangular

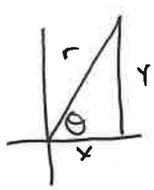
θ	r	x	y
$45°$	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{2\pi}{3}$	2	-1	$\sqrt{3}$
$45°$	4	$\frac{4}{\sqrt{2}}$	$\frac{4}{\sqrt{2}}$
$\frac{-\pi}{6}$	3	$\frac{3\sqrt{3}}{2}$	$-\frac{3}{2}$

Workspace



3. Graph the Polar Equation $r = 1 + \cos\theta$ for $0 \le \theta \le 360°$ in increments of $15°$ on a polar grid of your choice. After completing the graph by hand, use Geogebra or another graphing utility to check and compare your results.

4. a. Draw a right triangle in the first quadrant and appropriately label its sides x, y, and r.



b. Write sine and cosine ratio equations and solve for x and y.

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$

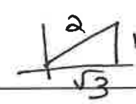
$y = r\sin\theta$ $x = r\cos\theta$

c. Using your polar equation $r = 1 + \cos\theta$ for $0 \le \theta \le 360°$, write parametric equations $x = f(\theta)$ and $y = g(\theta)$. (Hint: Use your information from part b. to get started.)

$r\sin\theta = \sin\theta(1 + \cos\theta)$ $r\cos\theta = \cos\theta(1 + \cos\theta)$

$y = \sin\theta(1 + \cos\theta)$ $x = \cos\theta(1 + \cos\theta)$

$\sin\frac{\pi}{6} = \frac{1}{2}$



$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

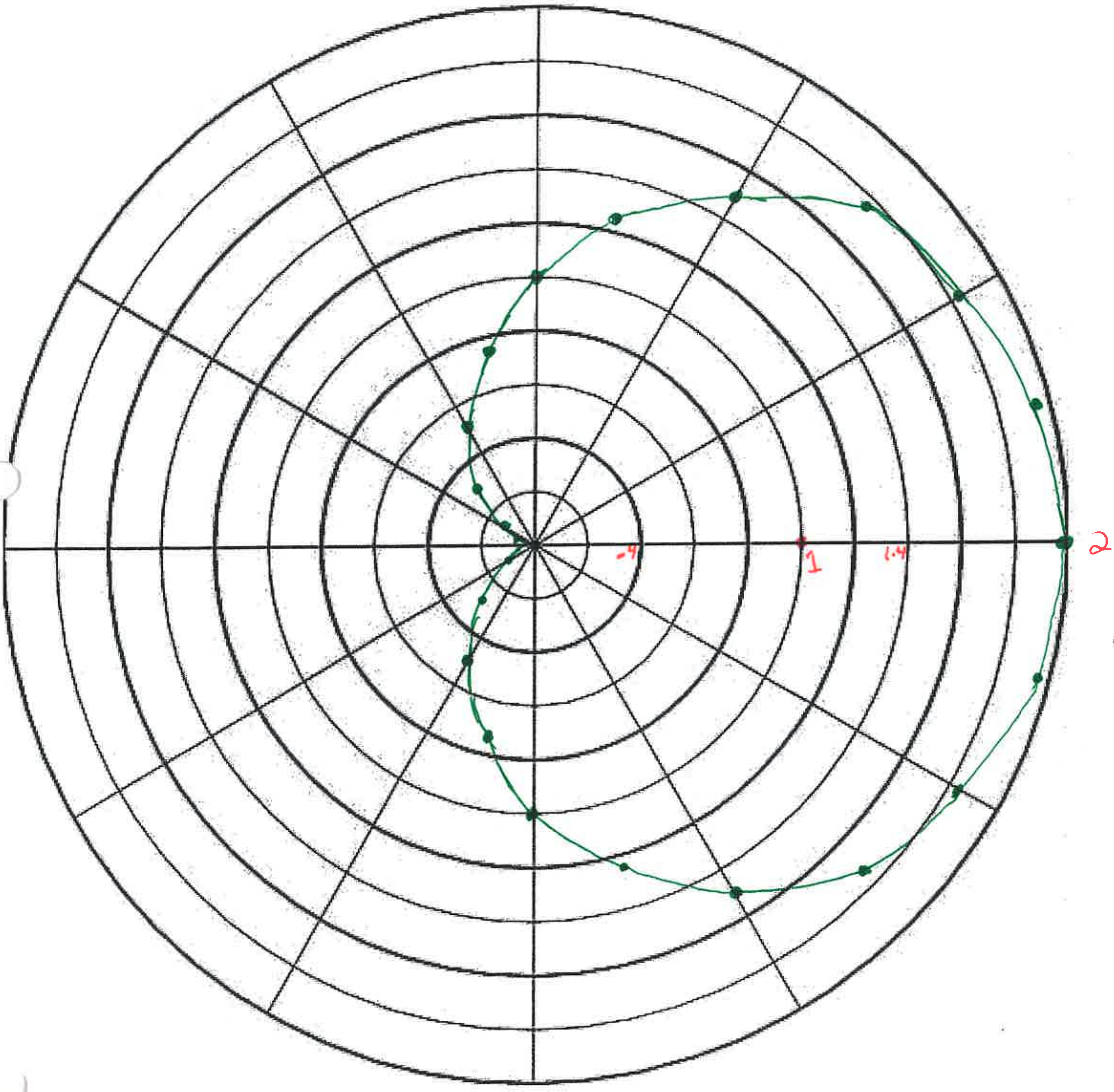
d. Determine the slope of the tangent line, $\frac{dy}{dx}$, of your polar equation when $\theta = \frac{\pi}{6}$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-\sin\theta) + (1+\cos\theta)(\cos\theta)}{\cos\theta(-\sin\theta) + (1+\cos\theta)(-\sin\theta)} = \frac{-\frac{1}{4} + \frac{\sqrt{3}}{2} + \frac{3}{4}}{-\frac{2\sqrt{3}}{4} - \frac{1}{2}}$$

$$\frac{2 + 2\sqrt{3}}{-2 + 2\sqrt{3}} = \frac{1 + \sqrt{3}}{\sqrt{3} - 1} = -1 \ m$$

e. Write an equation for the tangent line to $r = 1 + \cos\theta$ when $\theta = \frac{\pi}{6}$. Use a graphing utility to visualize your polar graph along with your tangent line.

$y(\frac{\pi}{6}) = (\frac{1}{2})(1 + \frac{\sqrt{3}}{2})$ $x(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}(1 + \frac{\sqrt{3}}{2})$ $y - (\frac{1}{2} + \frac{\sqrt{3}}{4}) = -1(x - (\frac{\sqrt{3}}{2} + \frac{3}{4}))$


.4
1
1.4
2

5. Let's examine a Circle with radius 5 centered at the origin.

a. Determine the area of this circle using geometrical methods.

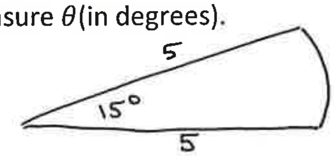
$A = \pi r^2$

$A = 25\pi$

b. Determine a Cartesian Equation for this circle, and then determine the area of this circle using calculus methods.

$r = 5$

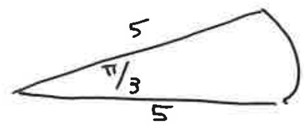
c. Draw a 15 degree sector of this circle. Determine the area of this sector. How would you determine the area of this sector with angle measure θ(in degrees).



$A = \frac{15°}{360°}(25\pi)$

$= \frac{25\pi}{24}$

d. Draw a $\frac{\pi}{3}$ radian sector. Determine the area of this sector...try to stay in radians. How would you determine the area of this sector with angle measure θ(in radians).



$A = \left(\frac{\pi/3}{2\pi}\right) 25\pi$

$= \frac{25\pi}{6}$

e. Write a formula for the area of a sector in terms of θ(in degrees) and r.

$A = \frac{\theta}{360}(\pi r^2)$

f. Write a formula for the area of a sector in terms of θ(in radians) and r.

$A = \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2} r^2 \theta$

g. Determine the polar equation for this circle. Set up an integral using what you see in part f. to determine that would determine the area of the circle. Evaluate the integral and verify.

$r = 5$

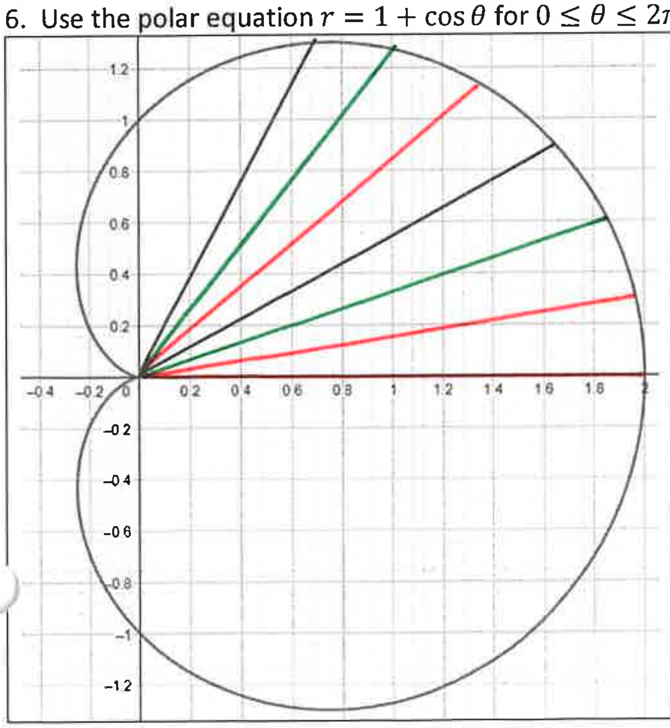
$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} 5^2 \, d\theta$

$= \frac{1}{2}\left[25\theta\right]_0^{2\pi} = \left(\frac{50\pi}{2}\right) - (0)$

$= 25\pi$

6. Use the polar equation $r = 1 + \cos\theta$ for $0 \leq \theta \leq 2\pi$ to answer the following questions.



a. Draw a few thin sectors representing the area of the polar equation.

b. Write an integral that will determine the area of this polar equation.

$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 \, d\theta$

c. Evaluate the integral on your calculator to determine the area. (Challenge: Evaluate the integral w/o your calculator. Hint: A half angle identity will be used.)

$A \approx 4.712$

many of these may have equivalent versions of correct solutions,

7. Use a graphing utility to check both forms of each equation.

	Polar Equation	Rectangular Equation
A Circle with Radius 5	$r = 5$	$x^2 + y^2 = 25$ or $y = \pm\sqrt{25 - x^2}$
Equation #2	$r = 1 + \cos\theta$	(This equation does not need to be solved explicitly for y.) $x^2 + y^2 = \sqrt{x^2 + y^2} + x$
Equation #3	$\theta = \frac{3\pi}{4}$	$y = -x$
Equation #4	$r = -8 \cdot cos\theta$	$y = \pm\sqrt{-x^2 - 8x}$
Equation #5	(This equation does not need to be solved explicitly for y.) $r^2 + 2r(\sin\theta - \cos\theta) = 0$	$(x-1)^2 + (y+1)^2 = 2$
Equation #6	$r \cdot \cos\theta = 4$	$x = 4$
Equation #7	$r\cos\theta = r^2\sin^2\theta$ $r = 0$ $r = \frac{\cos\theta}{\sin^2\theta}$	$x = y^2$
Equation #8	$r \cdot \sin\theta = -3$	$y = -3$
Equation #9	(This equation does not need to be solved explicitly for r.) $2r\cos\theta - 5(r\cos\theta)^3 = 1 + (r\cos\theta)(r\sin\theta)$	$2x - 5x^3 = 1 + xy$

Workspace

#2 $r \cdot r = (1 + \cos\theta) r$

$r^2 = r + r\cos\theta$

$x^2 + y^2 = \sqrt{x^2 + y^2} + x$

#5 $(x-1)^2 + (y+1)^2 = 2$

$(r\cos\theta - 1)^2 + (r\sin\theta + 1)^2 = 2$

$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta + 2r\sin\theta + 1 = 2$

$r^2(\cos^2\theta + \sin^2\theta) + 2r(-\cos\theta + \sin\theta) = 0$

$r^2 + 2r \cdot (\sin\theta - \cos\theta) = 0$

#4 $r = -8\cos\theta$

$r^2 = -8r\cos\theta$

$x^2 + y^2 = -8x$

$y = \pm\sqrt{-x^2 - 8x}$

$x^2 + y^2 = r^2$

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

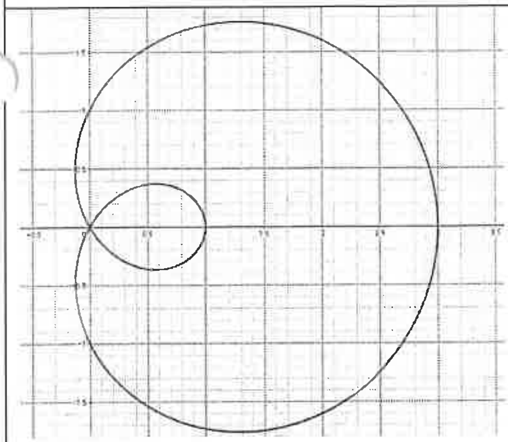
$r = \sqrt{x^2 + y^2}$

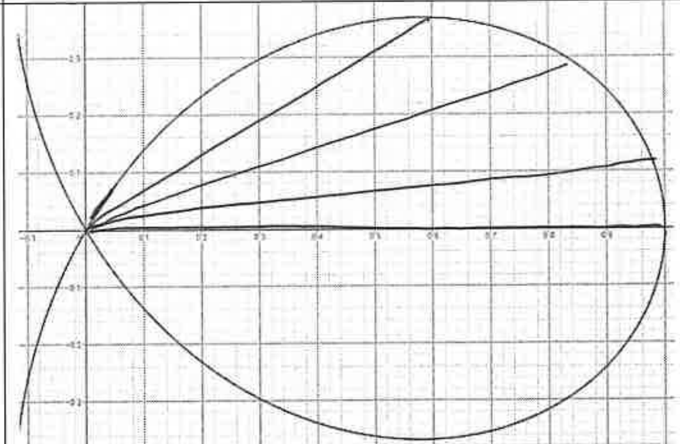
$y = r\sin\theta$

$x = r\cos\theta$

8. Determining the Area of a Portion of a Polar Curve.

Given: $r = 1 + 2\cos\theta,\ 0 \le \theta \le 2\pi$





θ	r
$\frac{\pi}{6}$	2.732
$\frac{\pi}{3}$	2
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	0
$\frac{5\pi}{6}$	-.732
π	-1
$\frac{7\pi}{6}$	-.732
$\frac{4\pi}{3}$	0
$\frac{3\pi}{2}$	1
$\frac{5\pi}{3}$	2
$\frac{11\pi}{6}$	2.732
2π	3

a. Determine the interval where the small "petal" occurs.

$\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$

b. Estimate the area of the small petal by using 8 subinterval sectors.

Full petal

$\frac{1}{2}\int_{2\pi/3}^{\pi} (1+2\cos\theta)^2\, d\theta \quad \times 2$

$2\pi/3 \to 0 \quad > .207$
$9\pi/12 \to .414 \quad > .573$
$10\pi/12 \to .732 \quad > .832$
$11\pi/12 \to .932 \quad > .966$
$3\pi/3 \to 1$

$\frac{1}{2}[.207^2 + .573^2 + .832^2 + .966^2]\,\pi/12$

$\approx 2\pi/12 \approx .523$

c. Write an integral that represents the area of the small petal. Evaluate this integral.

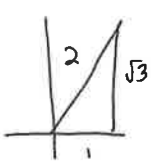
$\approx .544$

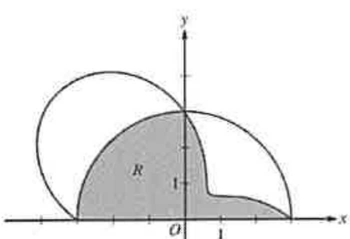
d. Write an integral that would represent the area of the cartiod not including the petal. Evaluate this area.

$\frac{1}{2}\int_{0}^{2\pi/3} (1+2\cos\theta)^2\, d\theta \times 2 \quad - .544$

≈ 8.337

9.





2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

a. $\frac{1}{2}\int_{0}^{\pi/2} (3 - 2\sin(2\theta))^2\, d\theta + \frac{\pi(3)^2}{4}$

≈ 9.708

b. $x = r\cos\theta$

$x = (3 - 2\sin(2\theta))\cos\theta$

$\left.\frac{dx}{d\theta}\right|_{x=\frac{\pi}{6}} \approx -2.366$

c. $d = 3 - 2\sin(2\theta) - 3$

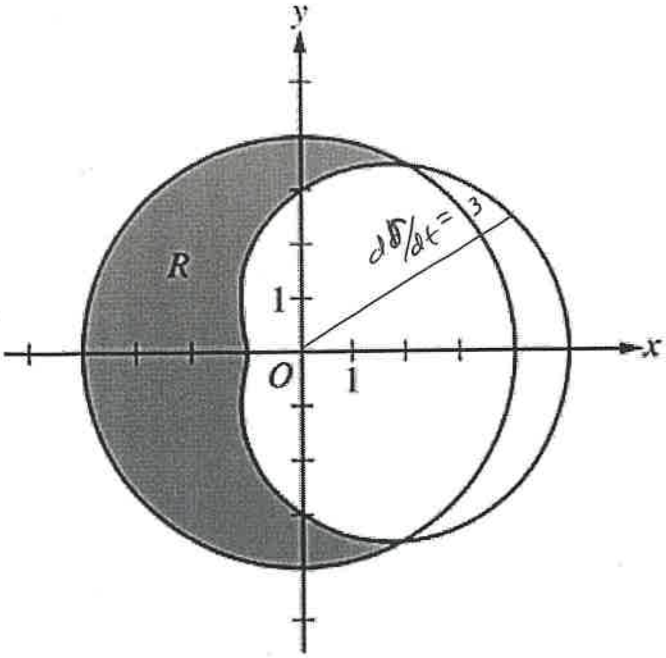
$\frac{dd}{d\theta} = -4\cos(2\theta)$

$\left.\frac{dd}{d\theta}\right|_{\theta=\frac{\pi}{3}} = 2$

$\frac{dr}{dt} = -4\cos(2\theta)\frac{d\theta}{dt}$

$= -4\cos(\frac{\pi}{3})(3) = -6$

c.



$$\frac{dr}{dt} = -2\sin\theta\,\frac{d\theta}{dt}$$

$$3 = -2\left(\frac{\sqrt{3}}{2}\right)\frac{d\theta}{dt}$$

$$-\frac{3}{\sqrt{3}} = \frac{d\theta}{dt}$$

$$-\frac{3}{\sqrt{3}}\ \frac{\text{RADIANS}}{\text{SECOND}}$$

or $-\sqrt{3}$

5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

$\pi/2$

a. $$R = \frac{1}{2}\int_{\pi/3}^{5\pi/3}\left[(4)^2\right]d\theta - \frac{1}{2}\int_{\pi/3}^{5\pi/3}(3 + 2\cos\theta)^2\,d\theta$$

$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$

b. $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$= 3\sin\theta + 2\sin\theta\cos\theta$

$x = r\cos\theta = (3 + 2\cos\theta)\cos\theta$

$\frac{dy}{d\theta} = 3\cos\theta + 2\sin\theta(-\sin\theta) + 2\cos^2\theta$

$x = 3\cos\theta + 2\cos^2\theta$

$\frac{dx}{d\theta} = -3\sin\theta + 4\cos\theta(-\sin\theta)$

$\left.\frac{dy}{d\theta}\right|_{\theta=\frac{\pi}{2}} = -2$

$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{2}} = -3$

$\frac{dy}{dx} = \frac{-2}{-3} = \frac{2}{3}$