$e^{x \ln 3}$

Intro: Integrate each. Check your answer using differentiation.

1. $\int x^2 dx$	2. $\int x\cos(x^2)\,dx$	3. $\int 3^x\,dx$

$\frac{1}{3}x^3 + c$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}\int 2x\cos(x^2)dx$

$\frac{1}{2}\int \cos u\,du$

$\frac{1}{2}\sin(u) + c$

$\frac{1}{2}\sin(x^2) + c$

$u = x\ln 3$

$du = \ln 3\,dx$

$\frac{1}{\ln 3}3^x + c$

Integration By Parts

One representation of the Product Rule.	Integration By Parts Formula
$\int \frac{d}{dx}(u\cdot v)\,dx = \int u\cdot\frac{dv}{dx}\,dx + \int v\cdot\frac{du}{dx}\,dx$ $u\cdot v = \boxed{\int u\,dv} + \int v\,du$ $\int u\,dv = u\cdot v - \int v\,du$	$\int u\,dv = u\cdot v - \int v\,du$

Integration by Parts Examples

4. $\int x\,sec^2x\,dx$	$\boxed{\text{5. } \int lnx\,dx}$	6. $\int x^2 sin\,3x\,dx$

4. $u = x \qquad dv = \sec^2 x\,dx$

$du = dx \qquad v = \tan x$

$\int x\cdot\sec^2 x\,dx = x\cdot\tan x - \boxed{\int \tan x\,dx}$

$= x\cdot\tan x + \ln|\cos x| + C$

5. $u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

$\int \ln x\,dx = x\cdot\ln x - \int x\cdot\frac{1}{x}\,dx$

$= x\ln x - x + C$

6. $u = x^2 \qquad dv = \sin(3x)\,dx$

$du = 2x\,dx \qquad v = -\frac{1}{3}\cos(3x)$

$\int x^2\sin(3x)\,dx = -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\int x\cos(3x)\,dx$

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\left[\frac{1}{3}x\sin(3x) - \frac{1}{3}\int\sin(3x)\,dx\right]$

$u = x \qquad dv = \cos(3x)\,dx$

$du = dx \qquad v = \frac{1}{3}\sin(3x)$

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{1}{9}\cos(3x) + C$

Integrate each using Integration By Parts. Check your answer using differentiation.

7. $\int x\cos x\,dx$	8. $\int e^x \cos x\,dx$	9. $\int x\sec x\tan x\,dx$	10. $\int x^5 e^{x^2}\,dx$	11. $\int x^2\,dx$

7.

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$\int x\cos x\,dx = x\cdot\sin x - \int \sin x\,dx$

$= x\cdot\sin x + \cos x + C$

8. $u = \cos x \qquad dv = e^x$

$du = -\sin x\,dx \qquad v = e^x$

$\int e^x \cos(x)\,dx = e^x\cdot\cos x + \int e^x \sin x\,dx$

$u = \sin x \qquad dv = e^x$

$du = \cos x\,dx \qquad v = e^x$

$\int e^x \cos x\,dx = e^x\cdot\cos x + \left[e^x \sin x - \int e^x \cos x\,dx\right]$

$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$

$\int e^x \cos x\,dx = \frac{1}{2}e^x\left[\cos x + \sin x\right] + C$

9. $u = x \qquad dv = \sec x\tan x\,dx$

$du = dx \qquad v = \sec x$

$\int x\cdot\sec(x)\tan(x)\,dx = x\cdot\sec x - \boxed{\int \sec x\,dx} \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$

$= x\cdot\sec(x) - \int \frac{\sec^2 x + \sec x\tan x}{\sec(x) + \tan(x)}\,dx$

$= x\cdot\sec(x) - \ln|\sec(x) + \tan x| + C$

10. $u = x^4 \qquad dv = xe^{x^2}$

$du = 4x^3\,dx \qquad v = \frac{1}{2}e^{x^2}$

$= \frac{1}{2}x^4 e^{x^2} - 2\int x^3 e^{x^2}\,dx$

$u = x^2 \qquad dv = xe^{x^2}$

$du = 2x\,dx \qquad v = \frac{1}{2}e^{x^2}$

$= \frac{1}{2}x^4 e^{x^2} - 2\left[\frac{1}{2}x^2 e^{x^2} - \int x e^{x^2}\,dx\right]$

$= \frac{1}{2}x^4 e^{x^2} - x^2 e^{x^2} + 1 e^{x^2} + C$

Using u-substitution. Remember that u-substitution is a tool in math to temporarily make a problem easier to view.

Examples and Practice

1. Solve $(2x+3)^2 + 2(2x+3) = 0$ LET $u = 2x+3$ $u^2 + 2u = 0$ $u(u+2) = 0$ $u = 0$ $u+2 = 0$ $2x+3 = 0$ $2x+5 = 0$ $x = \frac{-3}{2}$ $x = \frac{-5}{2}$	2. Factor $(5x+1)^2 - 3(5x+1) - 4$ LET $u = 5x+1$ $u^2 - 3u - 4$ $(u-4)(u+1)$ $(5x-3)(5x+2)$
3. Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ Find $\frac{dy}{dx}$ for $y = (5x+1)^6$ $\frac{dy}{dx} = 6(5x+1)^5 \cdot 5$	4. Find $\frac{dy}{dx}$ for $y = cos^2 x$ $\frac{dy}{dx} = 2\cos(x)\,(-\sin(x))$

5. Compare $\int_0^3 (x-1)^2 dx$ vs. $\int_{-1}^2 x^2 dx$

IN CLASS

6. $\int 3x^2\sqrt{x^3+5}\,dx$ LET $u = x^3+5$ $du = 3x^2 dx$ $\int u^{1/2}\,du$ $\frac{2}{3}u^{3/2}$ $= \frac{2}{3}(x^3+5)^{3/2} + c$	7. $\int \frac{1}{(1-6t)^4} dt$ LET $u = 1-6t$ $du = -6dt$ $-\frac{1}{6}\int u^{-4}\,du$ $= \frac{1}{18}u^{-3}$ $= \frac{1}{18}(1-6t)^{-3} + c$

8.

$$\int \frac{8x}{(4x^2+1)^2}\,dx$$

LET $u = 4x^2+1$

$du = 8x\,dx$

$\int u^{-2}\,du = -u^{-1} + c$

$= \frac{-1}{4x^2+1} + c$

9.

$$\int \sqrt{x}\,\sin\left(1+x^{\frac{3}{2}}\right)dx$$

$u = 1 + x^{3/2}$

$du = \frac{3}{2}x^{1/2}\,dx$

$\frac{2}{3}\int \sin u\,du$

$-\frac{2}{3}\cos u + c$

$= -\frac{2}{3}\cos(1+x^{3/2}) + c$

10.

$$\int x\cos(x^2)\,dx$$

LET $u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + c$

$= \frac{1}{2}\sin(x^2) + c$

11.

$$\int (1+\tan(\theta))^5\,\sec^2(\theta)\,d\theta$$

LET $u = 1+\tan\theta$

$du = \sec^2\theta\,d\theta$

$= \int u^5\,du$

$= \frac{1}{6}u^6 + c$

$= \frac{1}{6}(1+\tan\theta)^6 + c$

12.

$$\int \cos(x)\,\sin(\sin(x))\,dx$$

LET $u = \sin x$

$du = \cos x\,dx$

$= \int \sin u\,du$

$= -\cos u + c$

$= -\cos(\cos x) + c$

13.

FIX $\frac{1}{4}\left[\frac{2}{3}(1+2x)^{3/2} - 2(1+2x)^{1/2}\right] + C$

$$\int \frac{x}{\sqrt{1+2x}}\,dx$$

LET $u = 1+2x$

$x = \frac{u-1}{2}$

$du = 2\,dx$

$= \frac{1}{2}\int \frac{u-1}{2}\cdot\frac{1}{u}\,du$

$= \frac{1}{4}\int \frac{u-1}{\sqrt{u}}\,du = \frac{1}{4}\left[\int du \cancel{- \int \frac{1}{u}\,du}\right]$

MISTAKE

$= \frac{1}{4}\left[u \cancel{- \ln|u|}\right] = \frac{1}{4}\left[1+2x \cancel{- \ln|1+2x|}\right] + c$

14.

$$\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$$

LET $u = \pi x^{-1}$

$du = -\frac{\pi}{x^2}\,dx$

$-\frac{1}{\pi}\int \cos u\,du$

$= \frac{-1}{\pi}\sin u + c$

$= \frac{-1}{\pi}\sin\left(\frac{\pi}{x}\right) + c$

15.

$$\int \frac{x^2}{\sqrt{1-x}}\,dx$$

LET $u = 1-x$

$x = 1-u$

$du = -dx$

$-\int \frac{(1-u)^2}{u^{1/2}}\,du$

$= -\int \left(\frac{1}{u^{1/2}} - 2u^{1/2} + u^{3/2}\right)du$

$= -\left[2u^{1/2} - \frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]$

$= -2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + c$

Integrate each. ***Integration By Parts may not be your choice here.*** Check your answer using differentiation.

12. $\int \frac{7}{5x+1} dx$	13. $\int \left(\frac{\sin x}{\cos x}\right) dx$	14. Since $\frac{d}{dx}[x^2\sin(x)] = x^2\cos(x) + 2x\sin(x)$, does $\int (x^2\cos(x) + 2x\sin(x))\,dx = x^2\sin(x) + C$? Prove it!
15. $\int e^{5x}\,dx$	16. $\int \csc(x)\cot(x)dx$	17. $\int \frac{4x^2}{5x^3-11} dx$
18. $\int \cot x\,dx$		19. Since $\frac{d}{dx}\left[\frac{x^2-1}{3x}\right] = \frac{(3x)(2x)-(x^2-1)(3)}{(3x)^2} = \frac{x^2+1}{3x^2}$, does $\int \left(\frac{x^2+1}{3x^2}\right) dx = \frac{x^2-1}{3x} + C$? Prove it!

12. LET $u = 5x+1$

$du = 5dx$

$\frac{7}{5}\int \frac{1}{u}\,du$

$= \frac{7}{5}\ln|5x+1| + c$

13. $-\ln|\cos x| + c$

15. $\frac{1}{5}e^{5x} + c$

16. $-\csc(x) + c$

17. LET $u = 5x^3 - 11$

$du = 15x^2\,dx$

$\frac{4}{15}\int \frac{du}{u}$

$= \frac{4}{15}\ln|u| + c$

$= \frac{4}{15}\ln|5x^3-11| + c$

18. $\int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$

14. $\int x^2\cos(x)\,dx \; +2\int \cancel{2}x\sin(x)\,dx$

$u = x^2 \quad dv = \cos(x)dx$

$du = 2x\,dx \quad v = \sin(x)$

$\cancel{u = x \quad dv = \sin(x)dx}$

$\cancel{du = dx \quad v = -\cos x}$

$= x^2\sin(x) - 2\int x\sin(x)\,dx + 2\int x\sin(x)\,dx$

$= x^2\sin(x) + c$

$\frac{1}{3}x^{-2}$

19. $\frac{1}{3}\int dx + \frac{1}{3}\int \frac{1}{x^2}dx$

$= \frac{1}{3}x - \frac{1}{3}x^{-1}$

$= \frac{1}{3}\left[x - \frac{1}{x}\right]$

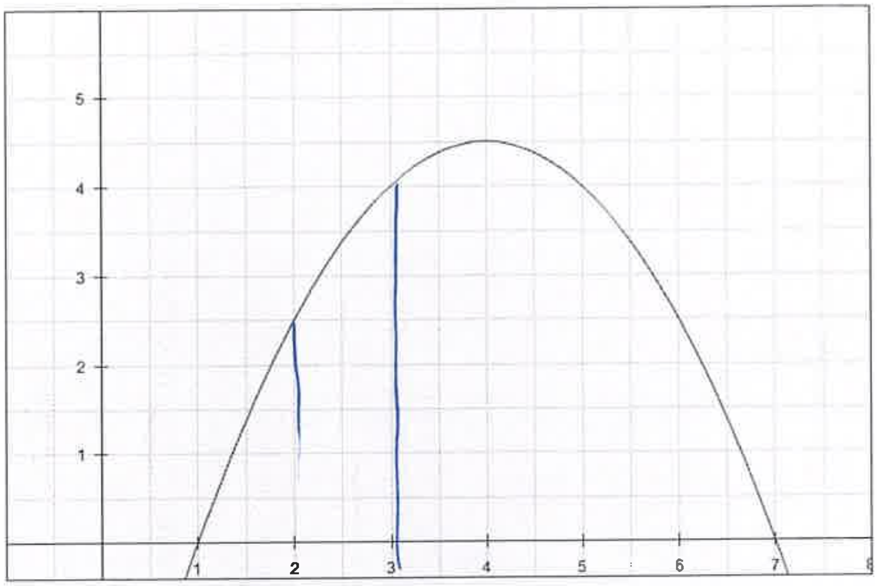
$= \frac{1}{3}\left[\frac{x^2}{x} - \frac{1}{x}\right]$

$= \frac{1}{3}\left[\frac{x^2-1}{x}\right]$

$= \frac{x^2-1}{3x} + c$

1. Given the quadratic function $f(x) = -.5(x-4)^2 + 4.5; \;\; 1 \le x \le 7$, approximate the Area Under the Curve (area between the curve and the x-axis) by using: a.k.a. $y = -\frac{1}{2}x^2 + 4x - 3.5$

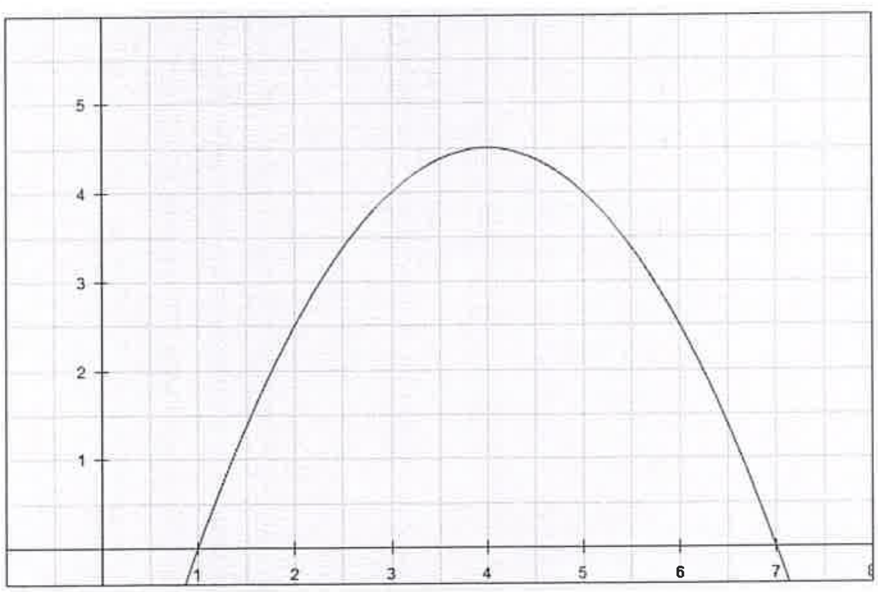
a. six rectangles created by matching the rectangles right corner to the function.



x	y
1	0
2	2.5
3	4
4	4.5
5	4
6	2.5
7	0

17.5

b. six rectangles created by matching the rectangles left corner to the function.



x	y
1	0
2	2.5
3	4
4	4.5
5	4
6	2.5
7	0

17.5

2. These are approximations. How could you get a more accurate answer?

MORE RECTANGLES

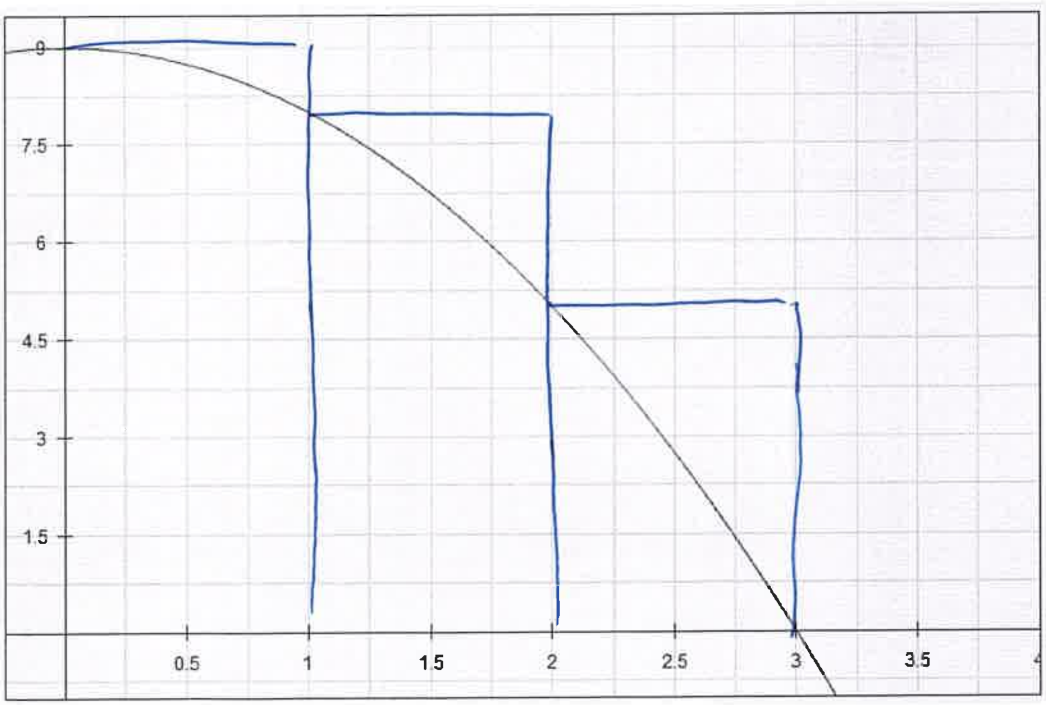
3. Given the quadratic function $(x) = -x^2 + 9;\ 0 \le x \le 3$, a.k.a. $y = -x^2 + 9$
approximate the Area Under the Curve (area between the curve and the x-axis) by using:

a. three rectangles created by matching the rectangles right corner to the function.

x	y
0	9
0.5	8.75
1	8
1.5	6.75
2	5
2.5	2.75
3	0

13

b. three rectangles created by matching the rectangles left corner to the function.



x	y
0	9
0.5	8.75
1	8
1.5	6.75
2	5
2.5	2.75
3	0

22

4. Is the approximation in part a. an over approximation or under approximation? How about part b.? Explain.

5. These are approximations. How could you get a more accurate answer?

6. Given the quadratic function $(x) = -x^2 + 9;\ 0 \le x \le 3$, a.k.a. $y = -x^2 + 9$
approximate the Area Under the Curve (area between the curve and the x-axis) by using:

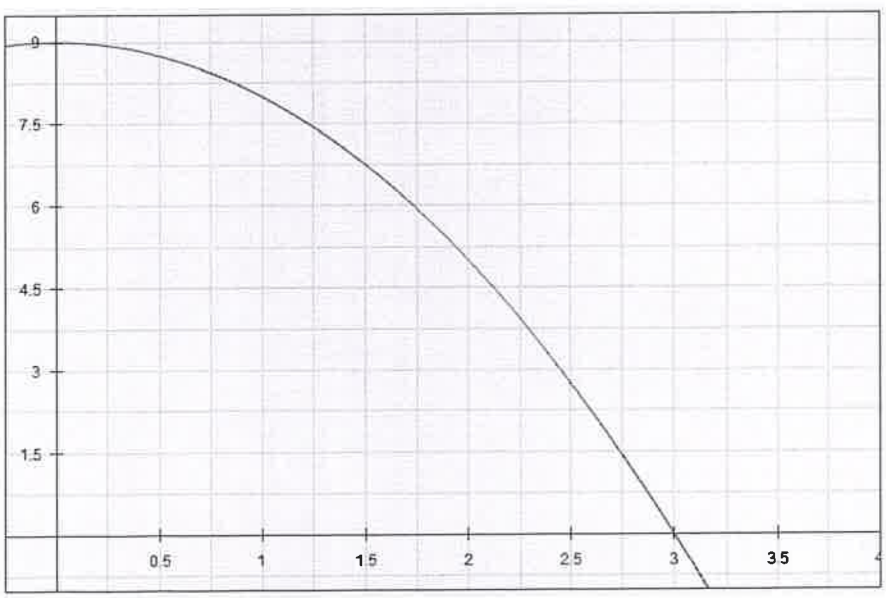
a. six rectangles created by matching the rectangles right corner to the function.

x	y
0	9
0.5	8.75
1	8
1.5	6.75
2	5
2.5	2.75
3	0

$\frac{31.25}{2}$

15.625

b. six rectangles created by matching the rectangles left corner to the function.



x	y
0	9
0.5	8.75
1	8
1.5	6.75
2	5
2.5	2.75
3	0

18.25

20.125

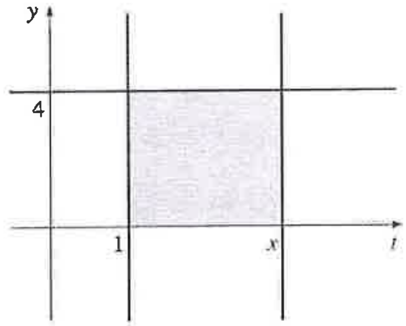
7. The answer to this area under the curve from $0 \le x \le 3$ is surprisingly a "nice" answer. What do you think it is?

GROUP WORK 1, SECTION 5.2

The Area Function

Recall that we can use the notation $\int_a^b f(t)\,dt$ to denote the area under the curve $f(t)$ between $t = a$ and $t = b$.

1. Consider the constant function $f(t) = 4$.



$\int_1^x 4\,dt$ + $4t\Big]_1^x$ = $4x - 4$

(a) Using geometry, compute $\int_1^2 f(t)\,dt$.

$(1)(4) = 4$

(b) Similarly compute $\int_1^3 f(t)\,dt$ and $\int_1^4 f(t)\,dt$.

$2(4) = 8$ $\quad$ $(3)(4) = 12$

(c) Using your answers to parts (a) and (b) as a guide, compute $\int_1^x f(t)\,dt$ for any $x \geq 1$.

$(x-1)(4)$

(d) We now define the *area function* $A(x) = \int_1^x f(t)\,dt$, $1 \leq x \leq 4$. What is $A(2)$? $A(2.5)$? $A(1)$? Write a general formula for $A(x)$.

$A(2) = 4$

$A(2.5) = (1.5)(4) = 6$

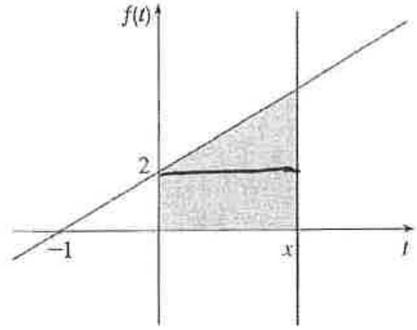
$A(1) = 0$

$A(x) = 4x - 4$

$\frac{2t^2}{2} + 2t\Big]_{-1}^{x}$

$x^2 + 2x - (1-2)$

2. Let $f(t) = 2t + 2$ for all t.



(a) Using geometry, compute $\int_0^2 f(t)\,dt$.

$2(2) + \frac{1}{2}(2)(2(2))$

$4 + 4 = 8$

(b) Similarly, compute $\int_0^4 f(t)\,dt$.

$4(2) + \frac{1}{2}(4)(2(4))$

$8 + 16 = 24$

(c) Using your answers to parts (a) and (b) as a guide, compute $\int_0^x f(t)\,dt$ for any $x \geq 0$.

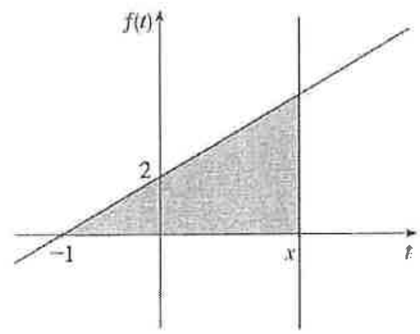
$2(x) + \frac{1}{2}(x)(2x)$

$2x + x^2$

(d) We now define another area function $B(x) = \int_0^x f(t)\,dt$. What is $B(2)$? $B(4)$? $B(0)$? Write a general formula for $B(x)$.

8 24 0

(e) We will now define a third area function $C(x) = \int_{-1}^x f(t)\,dt$ for any $x \geq -1$, as pictured below:



$x^2 + 2x + 1$

$(x+1)^2$

What is $C(2)$? $C(4)$? $C(-1)$? Write a general formula for $C(x)$.

$\frac{1}{2}(x+1)(2x+2)$

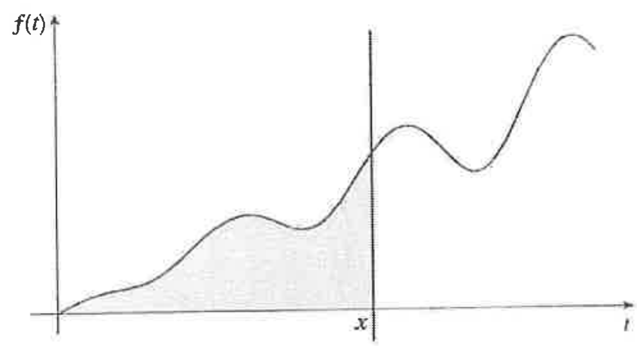
3. The Punchline:

We have now computed three different area functions. Fill in the blanks:

$A(x) =$ 4x − 4	$A'(x) =$ 4
$B(x) =$ x² + 2x	$B'(x) =$ 2x + 2
$C(x) =$ x² + 2x + 1	$C'(x) =$ 2x + 2

You should notice a very interesting fact about the derivatives of the area functions — a fundamentally beautiful property. What is it?

4. We are going to define one final function, $D(x) = \int_0^x 0.2t \sin(\cos(\sin t))\, dt$.

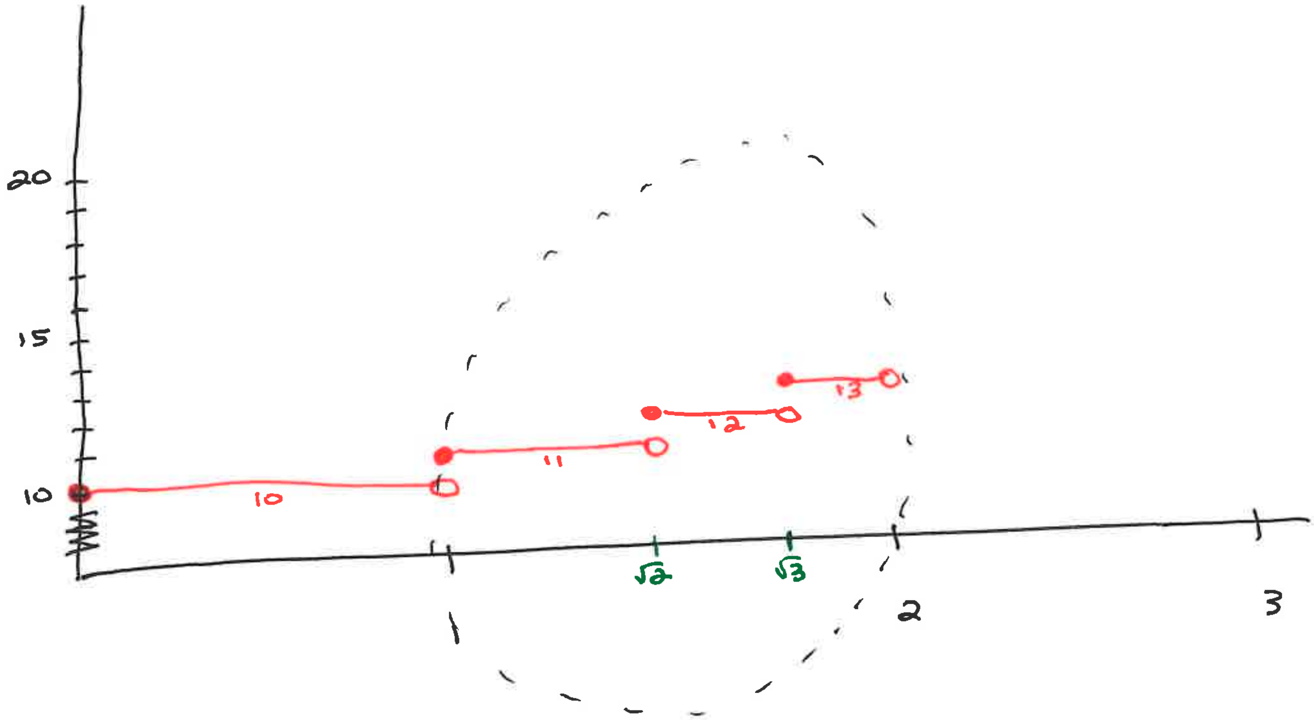


Don't worry about trying to find a simple formula for $D(x)$. But, using our amazing fact, fill in the last blank:

$\boxed{D'(x) = .2t \sin(\cos(\sin t))}$

Not only is the fact that we've discovered a surprising one; as we shall see in Section 5.4, it is also extremely useful.

3. Compute $\int_1^2 \left(10 + [\![x^2]\!]\right) dx$, where $[\![u]\!]$ is the greatest integer function defined in Section 2.3.
Hint: First draw the graph of $f(x) = 10 + [\![x^2]\!]$, $1 \le x \le 2$.



$(\sqrt{2} - 1)(11) + (\sqrt{3} - \sqrt{2})(12) + (2 - \sqrt{3})(13)$

≈ 11.854