Review – Linear Approximation and Tangent Lines

1. If the line tangent to the graph of the function f at the point $(2, 8)$ passes through the point $(-1, -1)$ then $f'(2) =$ 3

$\frac{8-(-1)}{2-(-1)}$

2. The y-intercept of the tangent line to the graph of $y = x^2 \ln x$ at $x = e$ is $-2e^2$

$f(x) = x^2 \ln x$

$f(e) = e^2$

$f'(x) = x^2\left(\frac{1}{x}\right) + \ln x(2x)$

$= x + 2x \ln x$

$f'(e) = e + 2e = 3e$

$y - e^2 = 3e(x - e)$

$y - e^2 = 3ex - 3e^2$

$y = \ldots -2e^2$

3. The slope of the line tangent to the curve $2xy + \sin y = 2\pi$ is

$2x\frac{dy}{dx} + y(2) + \cos(y)\frac{dy}{dx} = 0$

$(2x + \cos(y))\frac{dy}{dx} = 2y$

$\frac{dy}{dx} = \frac{2y}{2x + \cos(y)}$

4. The y-intercept of the line tangent to the graph of $y = \frac{x}{3x-2}$ at the point where $x = 1$ is 3

$f(x) = \frac{x}{3x-2}$ $\quad f(1) = 1$

$f'(x) = \frac{(3x-2)(1) - x(3)}{(3x-2)^2}$

$f'(1) = \frac{1-3}{1} = -2$

$y - 1 = -2(x-1)$

$y - 1 = -2x + 2$

$y = -2x + 3$

5. Let f be a differentiable function such that $f(3) = 5$ and $f'(3) = -2$. If the tangent line to the graph of f at $x = 3$ is used to find an approximation to a zero of f, that approximation is

$y - 5 = -2(x - 3)$

$-5 = -2x + 6$

$-11 = -2x$

$x = \frac{11}{2}$

6. Consider the differential equation $\frac{dy}{dx} = e^y(3x^2 - 6x)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$. Write the equation of the tangent line at $(1, 0)$ and use it to approximate $f(1.2)$.

$\left.\frac{dy}{dx}\right|_{(1,0)} = 1(3-6) = -3$

$y = -3(x-1)$

$f(1.2) \approx -3(1.2 - 1) = -.6$

7. For each, find an equation for the tangent line for function f when $x = 2$. Use this linearization to approximate the value of f when $x = 1.9$.

a. $f(x) = \frac{6}{x^2}$ $\quad 6x^{-2}$ $\quad f(2) = \frac{3}{2}$

$f'(x) = -\frac{12}{x^3}$ $\quad f'(2) = \frac{-12}{8} = \frac{-3}{2}$

$y - \frac{3}{2} = -\frac{3}{2}(x - 2)$

$f(1.9) \approx \frac{3}{2} - \frac{3}{2}\left(-\frac{1}{10}\right) = \frac{3}{2} + \frac{3}{20}$

b. $f(x) = x^5$ $\quad f(2) = 32$

$f'(x) = 5x^4$ $\quad f'(2) = 80$

$y - 32 = 80(x - 2)$

$f(1.9) = 32 + 80\left(-\frac{1}{10}\right)$

$= 24$

c. $f(x) = \sqrt{x}$ $\quad f(2) = \sqrt{2}$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(2) = \frac{1}{2\sqrt{2}}$

$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$

$f(1.9) = \sqrt{2} + \frac{1}{2\sqrt{2}}\left(-\frac{1}{10}\right)$

$= \sqrt{2} - \frac{1}{20\sqrt{2}}$

8. If $y = 2xy + 5$, find dy when $x = 1$ and $dx = .3$

$\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$

$x = 1$: $y = 2y + 5$, $y = -5$

$dy = 2x\,dy + 2y\,dx$

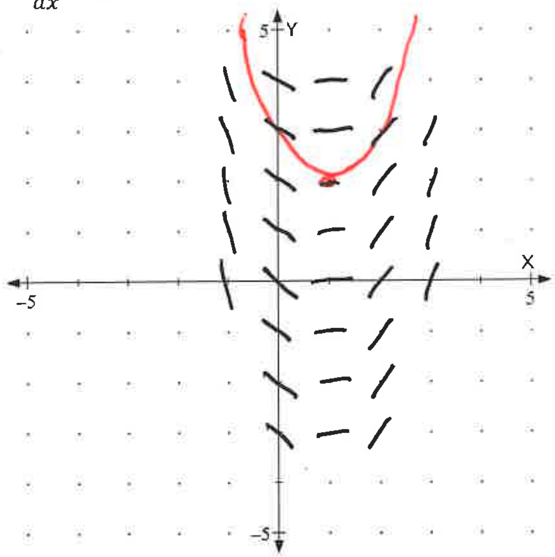
~~[illegible]~~

$dy = 2dy - 10(.3)$

$3 = dy$

Draw a slope field for each. Use enough points to get a good feel for how the slope field behaves. Answer any additional information with regards to the differential equation or its particular solutions.

1. $\frac{dy}{dx} = x - 1$



Graph the particular solution of this differential equation with the initial condition of $(1, 2)$.

Solve this differential equation with this same initial condition. Does your graph seem to match?

$$\frac{dy}{dx} = x - 1$$

$$dy = (x-1)\,dx$$

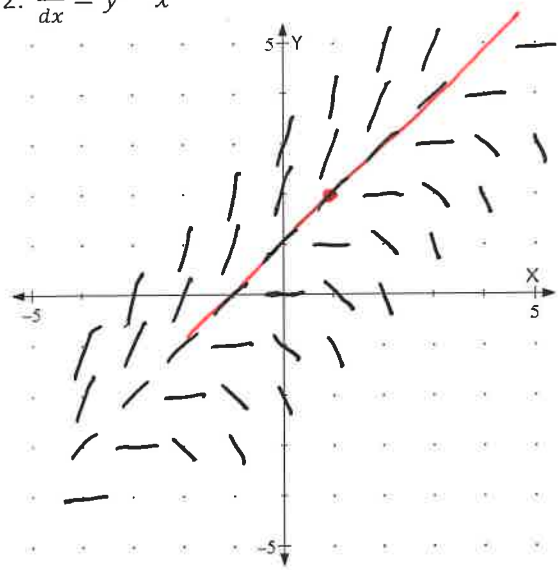
$$y = \frac{1}{2}x^2 - x + c$$

$$(1,2) \rightarrow 2 = \frac{1}{2} - 1 + c$$

$$\frac{5}{2} = c$$

$$y = \frac{1}{2}x^2 - x + \frac{5}{2}$$

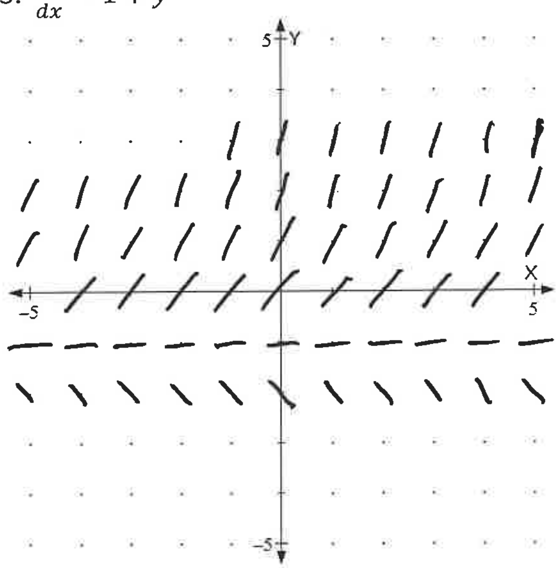
2. $\frac{dy}{dx} = y - x$



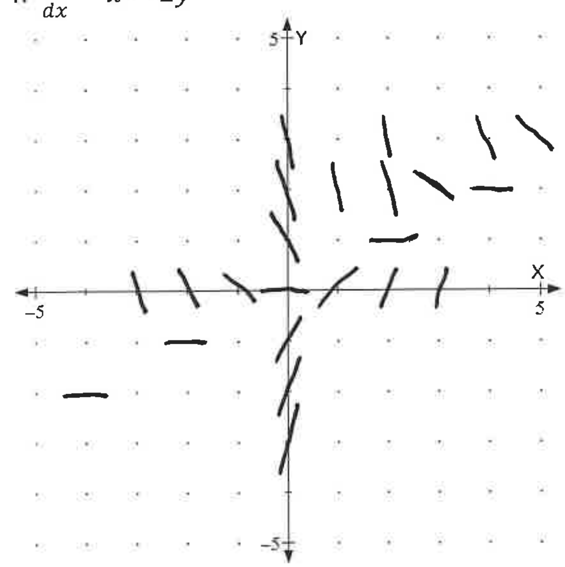
Graph the particular solution of this differential equation with the initial condition $f(1) = 2$.

Explain the issue when attempting to solve this differential equation.

3. $\frac{dy}{dx} = 1 + y$



4. $\frac{dy}{dx} = x - 2y$



Euler's Method of Approximation (Used to Approximate function values when given a differential equation)

The Idea of Euler's Method

Let's say we have a function with a constant slope of 5 with an initial condition of $f(0)) = 10$.

a. If the x-coordinate changes $+4$, how much would the y-coordinate change and what would be the new coordinate pair.

b. Now from that last point if the x-coordinate changes $+1.3$, how much would the y-coordinate change and what would be the new coordinate pair.

Use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of $f(x)$ when $x = 1.3$.

1. $\frac{dy}{dx} = x - 1$ and $f(1) = 2$

$f(1) = 2$

$f(1.1) = 2 + (1-1)(.1) = 2$

$f(1.2) = 2 + (1.1-1)(.1) = 2.01$

$f(1.3) = 2.01 + (1.2-1)(.1) = 2.03$

2. $\frac{dy}{dx} = y - x$ and $f(1) = 2$

$f(1) = 2$

$f(1.1) = 2 + (2-1)(.1) = 2.1$

$f(1.2) = 2.1 + (2.1-1.1)(.1) = 2.2$

$f(1.3) = 2.2 + (2.2-1.2)(.1) = 2.3$

Use Euler's Method with three equal increments to approximate the value of $f(x)$ when $x = 1.7$.

3. $\frac{dy}{dx} = 1 + y$ and $f(2) = 0$

$f(2) = 0$

$f(1.9) = 0 + (1+0)(-.1) = -.1$

$f(1.8) = -.1 + (1+-.1)(-.1) =$

$= -.1 + (\frac{9}{10})(\frac{-1}{10}) = -.19$

$f(1.7) = -.19 + (1+-.19)(-.1)$

$= -\frac{19}{100} + (\frac{81}{100})(\frac{-1}{10})$

$= -\frac{19}{100} + \frac{-81}{1000} = -.271$

4. $\frac{dy}{dx} = x - 2y$ and $f(2) = 1$

$f(2) = 1$

$f(1.9) = 1 + (2-2)(-.1) = 1$

$f(1.8) = 1 + (1.9-2)(-.1)$

$= 1 + .01 = 1.01$

$f(1.7) = 1.01 + (1.8-2.02)(-.1)$

$= 1.01 + .022 = 1.032$

5. For questions 1-4, determine the actual function value indicated...when you are able.

2 & 4 ARE INSEPERABLE DIF EQS, SO NOT ABLE

$\frac{dy}{dx} = x - 1$, $f(1) = 2$

1. $y = \frac{1}{2}x^2 - x + 2.5$

$y(1.7) = .5(1.3)^2 - (1.3) + 2.5 \approx 2.045$

EULER $f(1.7) \approx 2.03$

~~2.045 - 2.03~~

3. $\frac{1}{(1+y)}\,dy = dx$

$\ln|1+y| = x + c$

$|1+y| = e^{x+c}$

$|1+y| = Ce^{x}$

$1 + y = Ce^{x}$

$y = Ce^{x} - 1$

at $(2,0)$

$0 = Ce^{2} - 1$

$C = \frac{1}{e^2}$

$y = \frac{1}{e^2}e^{x} - 1$

$y = e^{-2+x} - 1$

$y(1.7) \approx -.259$

6. Determine the percent error for each of your approximations...when you are able.

1. $\left|\frac{2.045 - 2.03}{2.045}\right| \approx .0073$

3. $\left|\frac{-.259 - (-.271)}{-.259}\right| \approx .046$

7. Use a more accurate version of Euler's method to achieve less error. Prove that the error is less.

1. $\frac{dy}{dx} = x - 1$ AND $f(1) = 2$

USING $\Delta x = .05$

$f(1) = 2$

$f(1.05) = 2 + (1-1)(.05) = 2$

$f(1.1) = 2 + (1.05-1)(.05) = 2.0025$

$f(1.15) = 2.0025 + (1.1-1)(.05) = 2.0075$

$f(1.2) = 2.0075 + (1.15-1)(.05) = 2.015$

$f(1.25) = 2.015 + (1.2-1)(.05) = 2.025$

$f(1.3) = 2.025 + (1.25-1)(.05) = 2.0375$

$\frac{2.045 - 2.0375}{2.045} \approx .0037$