2024

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

   (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

   (b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

   (c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

   (d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

2. A particle moves along the x-axis so that its velocity at time $t \geq 0$ is given by $v(t) = \ln\left(t^2 - 4t + 5\right) - 0.2t$.

(a) There is one time, $t = t_R$, in the interval $0 < t < 2$ when the particle is at rest (not moving). Find t_R. For $0 < t < t_R$, is the particle moving to the right or to the left? Give a reason for your answer.

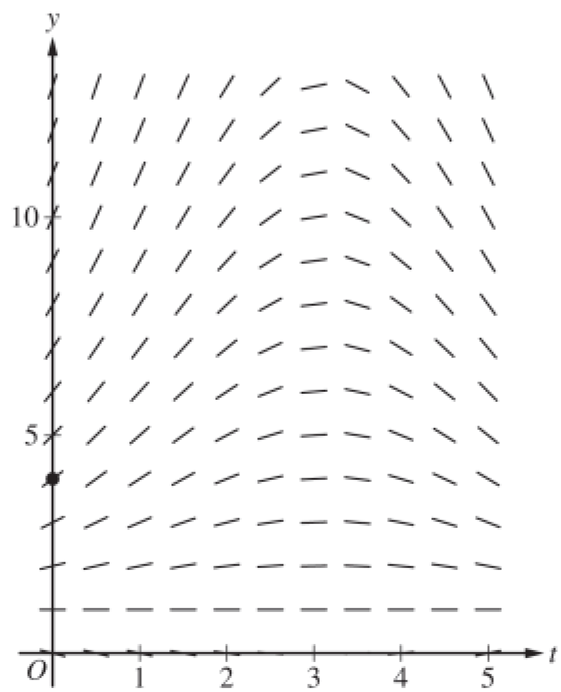
(b) Find the acceleration of the particle at time $t = 1.5$. Show the setup for your calculations. Is the speed of the particle increasing or decreasing at time $t = 1.5$? Explain your reasoning.

(c) The position of the particle at time t is $x(t)$, and its position at time $t = 1$ is $x(1) = -3$. Find the position of the particle at time $t = 4$. Show the setup for your calculations.

(d) Find the total distance traveled by the particle over the interval $1 \leq t \leq 4$. Show the setup for your calculations.

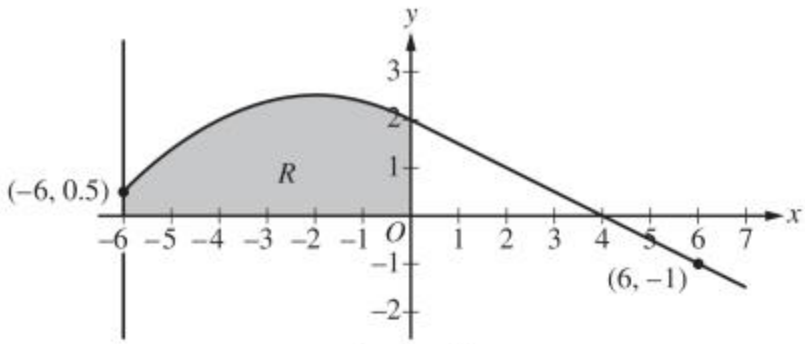
3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.



Graph of f

4. The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

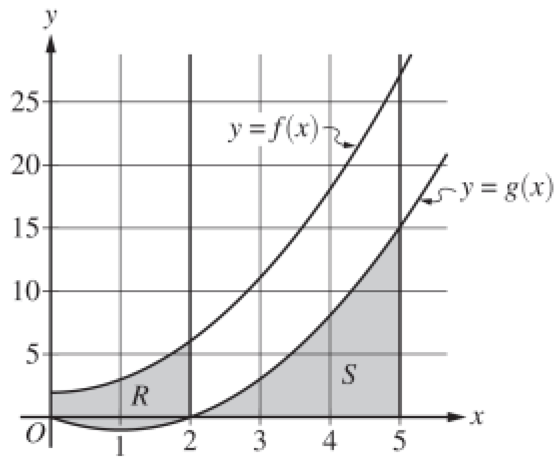
5. Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\dfrac{dy}{dx} = \dfrac{-2x}{3 + 4y}$.

(a) There is a point on the curve near $(2, 4)$ with x-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y-coordinate of this point.

(b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

(c) The curve intersects the positive x-axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

(d) For time $t \ge 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y-coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x-coordinate of the particle's position with respect to time?



6. The functions f and g are defined by $f(x) = x^2 + 2$ and $g(x) = x^2 - 2x$, as shown in the graph.

(a) Let R be the region bounded by the graphs of f and g, from $x = 0$ to $x = 2$, as shown in the graph. Write, but do not evaluate, an integral expression that gives the area of region R.

(b) Let S be the region bounded by the graph of g and the x-axis, from $x = 2$ to $x = 5$, as shown in the graph. Region S is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis is a rectangle with height equal to half its base in region S. Find the volume of the solid. Show the work that leads to your answer.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when region S, as described in part (b), is rotated about the horizontal line $y = 20$.

2023

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

   (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

   (b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

   (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

   Show the setup for your calculations.

   (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

2. Stephen swims back and forth along a straight path in a 50-meter-long pool for 90 seconds. Stephen's velocity is modeled by $v(t) = 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)$, where t is measured in seconds and $v(t)$ is measured in meters per second.

(a) Find all times t in the interval $0 < t < 90$ at which Stephen changes direction. Give a reason for your answer.

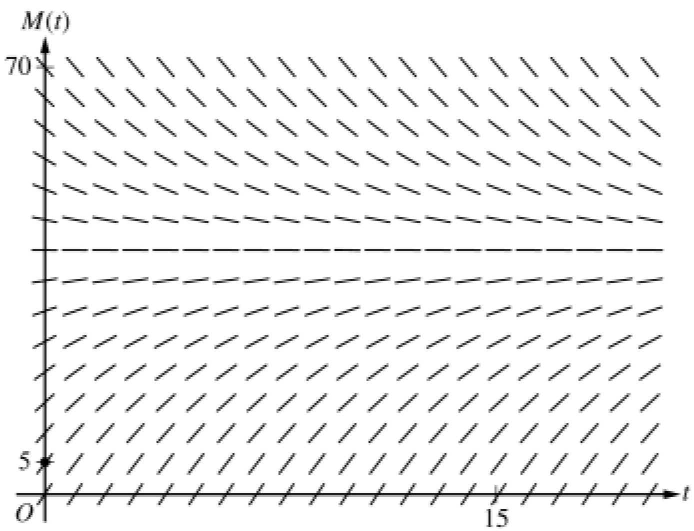
(b) Find Stephen's acceleration at time $t = 60$ seconds. Show the setup for your calculations, and indicate units of measure. Is Stephen speeding up or slowing down at time $t = 60$ seconds? Give a reason for your answer.

(c) Find the distance between Stephen's position at time $t = 20$ seconds and his position at time $t = 80$ seconds. Show the setup for your calculations.

(d) Find the total distance Stephen swims over the time interval $0 \le t \le 90$ seconds. Show the setup for your calculations.

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

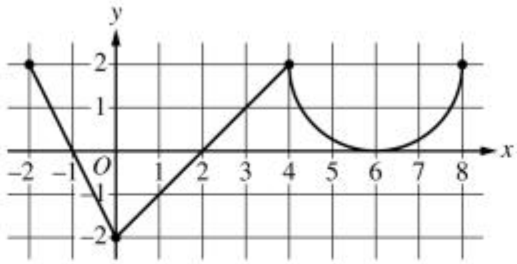
(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.



(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.



Graph of f'

4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

5. The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

6. Consider the curve given by the equation $6xy = 2 + y^3$.

(a) Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$.

(b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ unit per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

2022

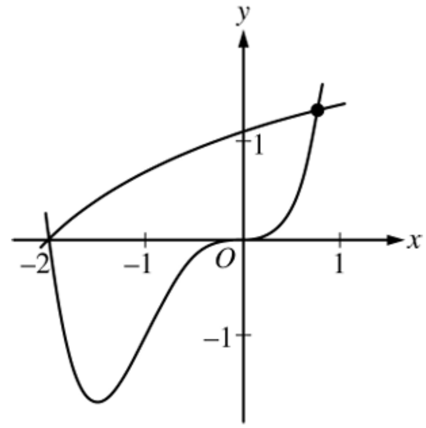
1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.



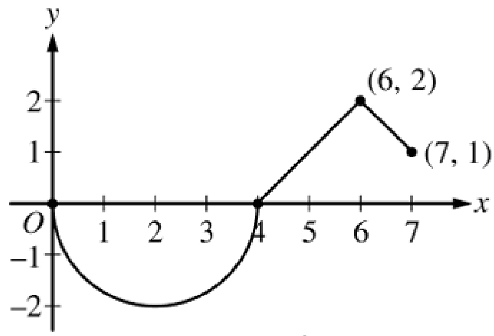
2. Let f and g be the functions defined by $f(x) = \ln(x + 3)$ and $g(x) = x^4 + 2x^3$. The graphs of f and g, shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.

(a) Find the area of the region enclosed by the graphs of f and g.

(b) For $-2 \le x \le B$, let $h(x)$ be the vertical distance between the graphs of f and g. Is h increasing or decreasing at $x = -0.5$? Give a reason for your answer.

(c) The region enclosed by the graphs of f and g is the base of a solid. Cross sections of the solid taken perpendicular to the x-axis are squares. Find the volume of the solid.

(d) A vertical line in the xy-plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of 7 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$.



Graph of f'

3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

   (a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

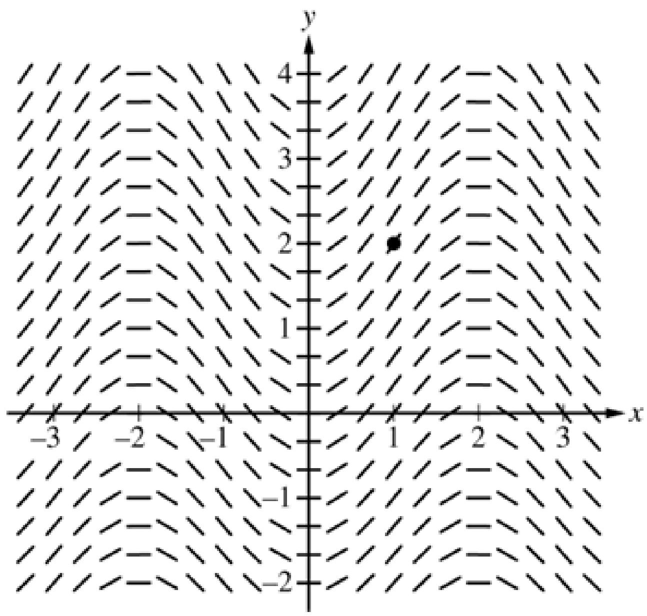
   (b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

   (c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

   (d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

6. Particle P moves along the x-axis such that, for time $t > 0$, its position is given by $x_P(t) = 6 - 4e^{-t}$.

Particle Q moves along the y-axis such that, for time $t > 0$, its velocity is given by $v_Q(t) = \frac{1}{t^2}$. At time $t = 1$, the position of particle Q is $y_Q(1) = 2$.

(a) Find $v_P(t)$, the velocity of particle P at time t.

(b) Find $a_Q(t)$, the acceleration of particle Q at time t. Find all times t, for $t > 0$, when the speed of particle Q is decreasing. Justify your answer.

(c) Find $y_Q(t)$, the position of particle Q at time t.

(d) As $t \to \infty$, which particle will eventually be farther from the origin? Give a reason for your answer.

2021

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

   (a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

   (b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r)\, dr$. Approximate the value of $2\pi \int_0^4 r f(r)\, dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

   (c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

   (d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

2. A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin\left(t^{1.5}\right)$ for $0 \le t \le \pi$. At time $t = 0$, particle P is at position $x = 5$.

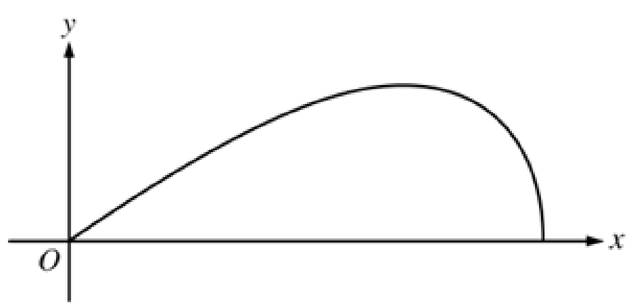
A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \le t \le \pi$. At time $t = 0$, particle Q is at position $x = 10$.

(a) Find the positions of particles P and Q at time $t = 1$.

(b) Are particles P and Q moving toward each other or away from each other at time $t = 1$? Explain your reasoning.

(c) Find the acceleration of particle Q at time $t = 1$. Is the speed of particle Q increasing or decreasing at time $t = 1$? Explain your reasoning.

(d) Find the total distance traveled by particle P over the time interval $0 \le t \le \pi$.

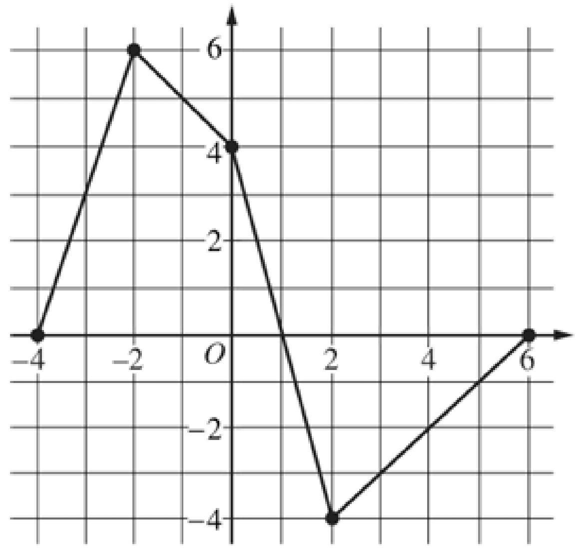


3. A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?



Graph of f

4. Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

5. Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$.

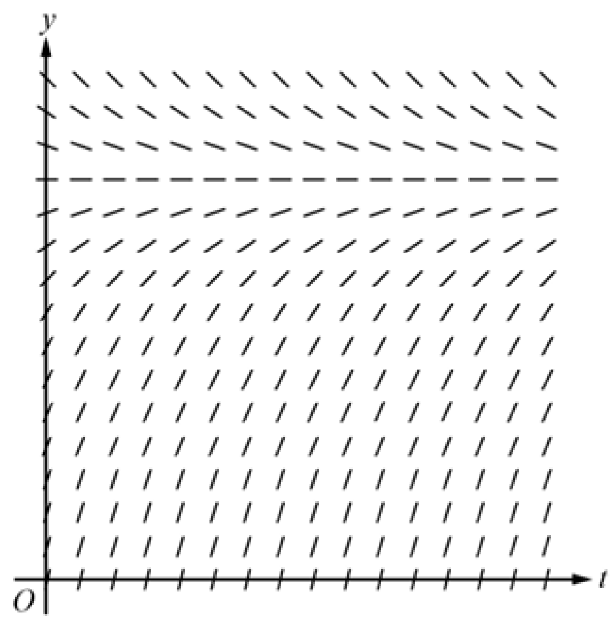
(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

6. A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.



(b) Using correct units, interpret the statement $\lim_{t \to \infty} A(t) = 12$ in the context of this problem.

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ with initial condition $A(0) = 0$.

2019

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

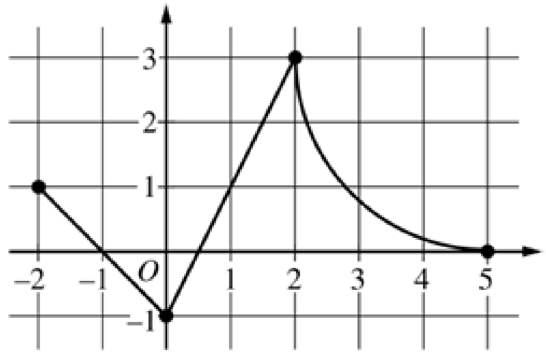
t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	–29	55	48

2. The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

(a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour.

(b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\,dt$.

(c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.



Graph of f

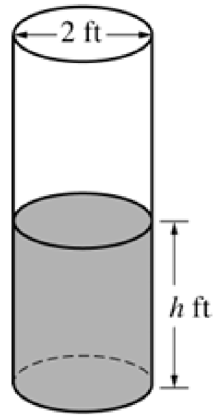
3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

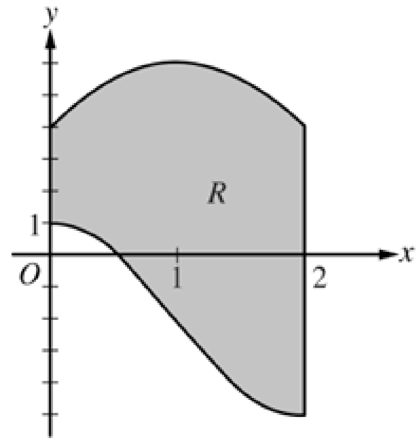


4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.



5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x - 1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x + 3}$. Find the volume of the solid.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \frac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

2018

1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

(d) For $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

2. A particle moves along the x-axis with velocity given by $v(t) = \dfrac{10\sin\left(0.4t^2\right)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

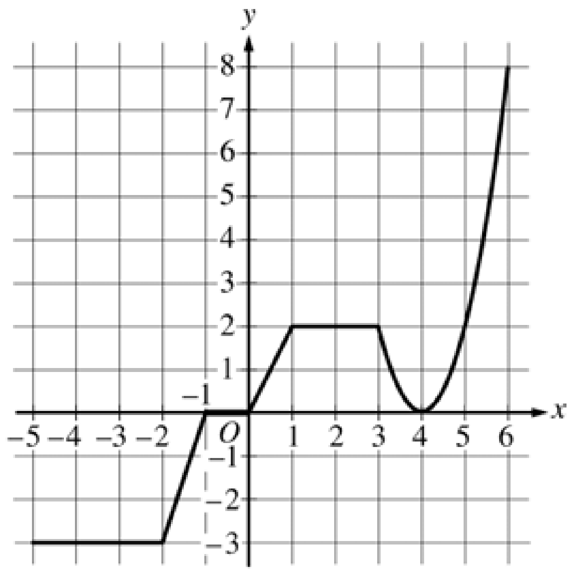
The particle is at position $x = -5$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 3$.

(b) Find the position of the particle at time $t = 3$.

(c) Evaluate $\int_0^{3.5} v(t)\,dt$, and evaluate $\int_0^{3.5} \left|v(t)\right|\,dt$. Interpret the meaning of each integral in the context of the problem.

(d) A second particle moves along the x-axis with position given by $x_2(t) = t^2 - t$ for $0 \le t \le 3.5$. At what time t are the two particles moving with the same velocity?



Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1 + x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

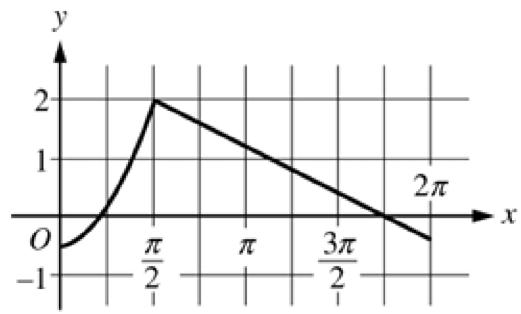
5. Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \le x \le \pi$.

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

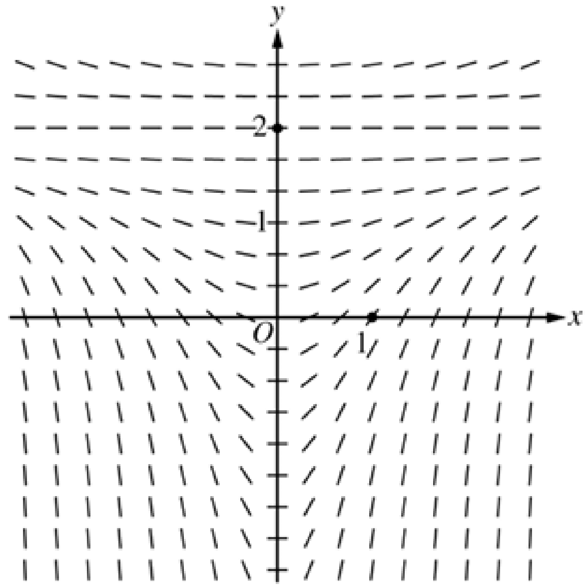
(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim\limits_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

   (a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

   (b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

   (c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

   (d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

2. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \leq 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \leq 12,$$

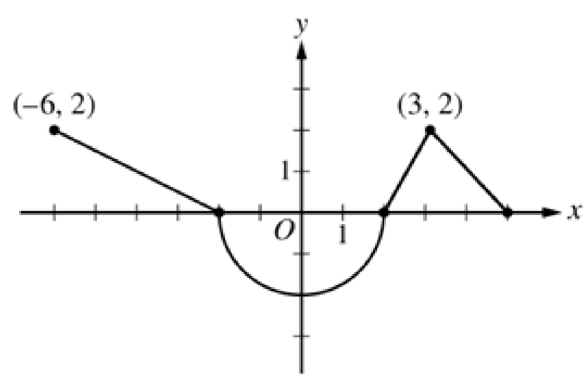
where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

(b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

(c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

(d) How many pounds of bananas are on the display table at time $t = 8$?



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is f increasing? Justify your answer.

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

5. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.

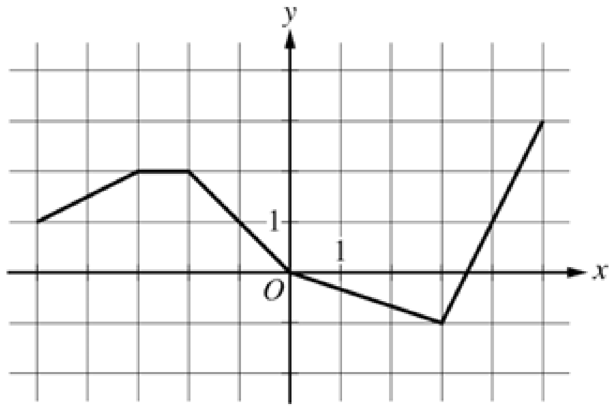
(a) For $0 \le t \le 8$, when is particle P moving to the left?

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

(d) Find the position of particle Q the first time it changes direction.

x	$g(x)$	$g'(x)$
–5	10	–3
–4	5	–1
–3	2	4
–2	3	1
–1	1	–2
0	0	–3



Graph of h

6. Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x.

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

(d) Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.